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ON THE P1 POWELL-SABIN DIVERGENCE-FREE FINITE ELEMENT FOR THE STOKES EQUATIONS*

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Dedicated to Professor Junzhi Cui on the occasion of his 70th birthday

Abstract

The stability of the P_1-P_0 mixed-element is established on general Powell-Sabin triangular grids. The piecewise linear finite element solution approximating the velocity is divergence-free pointwise for the Stokes equations. The finite element solution approximating the pressure in the Stokes equations can be obtained as a byproduct if an iterative method is adopted for solving the discrete linear system of equations. Numerical tests are presented confirming the theory on the stability and the optimal order of convergence for the P_1 Powell-Sabin divergence-free finite element method.

Mathematics subject classification: 65M60, 65N30, 76D07.
Key words: Powell Sabin triangles, Mixed finite elements, Stokes, Divergence-free element.

1. Introduction

A natural finite element method for the Stokes equations would be the P_k-P_{k-1} mixed element which approximates the velocity by continuous P_k piecewise-polynomials and approximates the pressure by discontinuous P_{k-1} piecewise-polynomials. One advantage of the element is to preserve the incompressibility condition of incompressible fluids, i.e., the discrete velocity is also divergence-free pointwise [2, 3, 18, 28, 29, 34]. Another advantage is its simplicity in computation that the mixed element can be reduced to the standard C_0 finite element solving Laplace equations, as the discrete pressure is a byproduct when an iterative method is used for the linear system of discrete equations. A fundamental study on the method was done by Scott and Vogelius in 1983 [28, 29] that the method is stable and consequently of the optimal order on 2D triangular grids for any $k \geq 4$, provided that the grids have no nearly-singular vertex (a vertex is called singular if all edges meeting at the vertex form two cross lines.) For $k \leq 3$, Scott and Vogelius showed the method would not be stable for general triangular grids, in [28, 29]. Nevertheless, for low order elements, ($k < 4$), the P_k-P_{k-1} element can still be stable if the underlying triangulations are of certain types. The stability is shown for the Hsieh-Clough-Tocher triangles with $k \geq 2$ [24], for the quadrilateral-triangulations with $k = 2$ [2], for the uniform criss-cross grids with $k = 1$ [25] and for the 3D Hsieh-Clough-Tocher tetrahedral grids with $k \geq 3$ [35].

To establish the convergence of the finite element solution for the pressure, a uniform (independent of the grid size h) inf-sup condition, (cf. (3.7)), known as LBB condition (cf. [6]), is usually required. For example, when a nearly-singular vertex approaches to singular (cf.

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[28, 29]), the inf-sup constant approaches to zero for the mixed element space P_k-P_{k-1}. Nevertheless, when the vertex becomes a singular one, the inf-sup constant would jump back from zero to a regular one, because the extra spurious pressure mode is filtered out automatically in the divergence-free element method. This is exactly the situation for the P_1-P_0 mixed element on Powell-Sabin grids. To be precise, here the discrete space for the pressure is the divergence of C_0-P_1 conforming element space, a proper subspace of the C_{-1}-P_0 space on the grid. Via the macro-element technique [6, 31] and a local L^2-orthogonal decomposition (see (3.14) below), we prove the inf-sup condition for the P_1-P_0 mixed element on Powell-Sabin grids. The stability of this P_1 Powell-Sabin element has not been fully studied previously, but the element was used in computation, cf. [8]. We note that the name of divergence-free mixed element is used often for non-conforming [4, 21, 32] or discontinuous Galerkin methods [7, 10, 14, 17], or even for weakly divergence-free methods [33]. But here, the discrete solutions for the velocity are truly divergence-free, including the points on the inter-element boundary.

The rest of the paper is divided into the following sections. In Section 2, we define the divergence-free finite element method and prove the uniqueness of such finite element solution. We will show how to reduce the P_k-P_{k-1} mixed element to the C_0-P_k element, and how to apply the classic iterated penalty method to solve the discrete, but positive definite, linear systems of equations. In Section 3, we will show the inf-sup condition for the Powell-Sabin element. In Section 4 we provide numerical tests using the Powell-Sabin P_1 divergence-free element.

2. The Divergence-free Finite Element

In this section, we shall define the divergence-free finite elements for the stationary Stokes equations. The resulting linear systems of equations, by such elements, are shown to have a unique solution. The classic iterated penalty method is introduced, which solves the discrete linear system and generates the discrete pressure solution as a byproduct.

We consider the stationary Stokes equations: Find functions $\mathbf{u}$ (the fluid velocity) and p (the pressure) on a 2D polygonal domain $\Omega \subset \mathbf{R}^2$ such that

$$\begin{aligned} -\Delta\mathbf{u} + \nabla p = \mathbf{f} \qquad & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \qquad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \qquad & \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\mathbf{f}$ is the body force. Via the integration by parts, we get a variational problem for the Stokes equations: Find $\mathbf{u} \in H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega) := L^2(\Omega)/C = \{p \in L^2 \mid \int_\Omega p = 0\}$ such that

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\mathbf{f},\mathbf{v}) \quad & \forall \mathbf{v} \in H_0^1(\Omega)^2, \\ b(\mathbf{u},q) = 0 \quad & \forall q \in L_0^2(\Omega). \end{aligned} \tag{2.2}$$

Here $H_0^1(\Omega)^2$ is the subspace of the Sobolev space $H^1(\Omega)^2$ (cf. [9]) with zero boundary trace, and

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &= \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}\, dx, \\ b(\mathbf{v},p) &= -\int_\Omega \operatorname{div}\mathbf{u}\, p\, dx, \\ (\mathbf{f},\mathbf{v}) &= \int_\Omega \mathbf{f}\ \mathbf{v}\, dx. \end{aligned}$$