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EXPONENTIAL CONVERGENCE OF SAMPLE AVERAGE APPROXIMATION METHODS FOR A CLASS OF STOCHASTIC MATHEMATICAL PROGRAMS WITH COMPLEMENTARITY CONSTRAINTS [*1)]

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Abstract

In this paper, we propose a Sample Average Approximation (SAA) method for a class of Stochastic Mathematical Programs with Complementarity Constraints (SMPCC) recently considered by Birbil, Gürkan and Listes [3]. We study the statistical properties of obtained SAA estimators. In particular we show that under moderate conditions a sequence of weak stationary points of SAA programs converge to a weak stationary point of the true problem with probability approaching one at exponential rate as the sample size tends to infinity. To implement the SAA method more efficiently, we incorporate the method with some techniques such as Scholtes' regularization method and the well known smoothing NCP method. Some preliminary numerical results are reported.

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Key words: Stochastic mathematical programs with complementarity constraints, Sample average approximation, Weak stationary points, Exponential convergence.

1. Introduction

In this paper, we investigate the following Stochastic Mathematical Programs with Complementarity Constraints (SMPCC)

$$\begin{array}{ll} \min\limits_{x\in\mathcal{X}} & \mathbb{E}[f(x,\xi(\omega))] \\ \text{s.t.} & \mathbb{E}[F(x,\xi(\omega))]\geq 0,\ \mathbb{E}[G(x,\xi(\omega))]\geq 0, \\ & 0\leq \mathbb{E}[F(x,\xi(\omega))]\perp \mathbb{E}[G(x,\xi(\omega))]\geq 0, \end{array} \tag{1.1}$$

where $f:\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}$, $F:\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^m$, $G:\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^m$ are twice continuously differentiable with respect to x for almost every ξ and continuous with respect to ξ, $\mathcal{X}\subset\mathbb{R}^n$ is a compact subset of $\mathbb{R}^n$, $\xi:\Omega\to\mathbb{R}^k$ is a vector of random variables defined on a probability space $(\Omega,\mathcal{F},P)$, $\mathbb{E}$ denotes the mathematical expectation.

The SMPCC model (1.1) was first considered by Birbil, Gürkan and Listes [3] and it is a natural extension of deterministic MPEC models [6]. The primary motivation for the model (1.1) is that the objective and constraint functions may involve some random data which reflect uncertainties in practical problems. For instance, in Stackelberg leader follower model [6], if

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players have to make a decision before the realization of uncertain demand, then each player has to consider the expected profit rather than the profit in a particular demand scenario. Subsequently the model can be reformulated as (1.1).

Birbil, Gürkan and Listes [3] applied the well known Sample Path (SP) method [9, 10] to solve (1.1). The basic idea of SP is to use computer simulation to approximate functions which are not observable. In this context, the expected value of functions in the objective and constraints of (1.1) are either not observable or very costly to be integrated out. Birbil, Gürkan and Listes [3] used simulation based average to construct successive approximate MPEC problems and showed that under some stability conditions the stationary points of approximate MPEC problems converge to their counterpart almost surely. More recently, Bastin, Cirillo and Toint [2] extended the discussion to investigate the convergence of stationary points for a broader class of stochastic optimization problems.

In this paper, we apply well known sample average approximation method to solve (1.1). Specifically, we consider an independent identically distributed (i.i.d) sample of $\xi(\omega)$, which is denoted by $\xi^1, ..., \xi^N$, and use the following *Sample Average Approximation* (SAA) problem to approximate the true problem (1.1):

$$\begin{array}{ll} \min\limits_{x\in\mathcal{X}} & \hat{f}_N(x) \\ \text{s.t.} & 0 \leq \hat{F}_N(x) \perp \hat{G}_N(x) \geq 0, \end{array} \tag{1.2}$$

where $\hat{f}_N(x) = \dfrac{1}{N}\sum_{i=1}^N f(x,\xi^i)$, and $\hat{F}_N(x) = \dfrac{1}{N}\sum_{i=1}^N F(x,\xi^i)$, $\quad \hat{G}_N(x) = \dfrac{1}{N}\sum_{i=1}^N G(x,\xi^i)$. SAA methods are essentially the same as SP methods and they have been extensively investigated in stochastic optimization. See recent work [1, 5, 16, 18]. More recently SAA methods have been applied to two stage stochastic mathematical programs with equilibrium constraints [17, 19, 21, 7] and various convergence results have been established.

In this paper, we analyze convergence of weak stationary points of SAA problem (1.2). This is motivated by the fact that due to the combinatorial nature, one may be more likely to obtain a stationary point than a local or global minimizer for MPEC problems. Consequently the notion of weak stationary point which was introduced by Scheel and Scholtes [12] is very relevant, indeed it has been well accepted and studied. The focus of this paper is on the convergence rate of weak stationary points of the SAA problem (1.2). Under some moderate conditions, we show the exponential convergence for weak stationary point of the SAA problems as sample size N tends to infinity. This result has significantly strengthened recent results of SAA methods for model (1.1).

The rest of this paper are organized as follows. In Section 2, we introduce some definitions about stationary points and present some preliminary results. In Section 3, we show that under some moderate stability conditions which are widely used in MPEC literature, the weak stationary points of SAA problem (1.2) converge to its counterpart of the true problem as the sample size tends to infinity. In Section 4, we discuss numerical implementation of the SAA method, and finally in Section 5 we report some preliminary test results.

2. Preliminaries

In this section, we recall some basic notions and definitions in mathematical programs with complementarity constraints. For simplicity of notation, let $\bar{f}(x) = \mathbb{E}[f(x,\xi(\omega))]$, $\bar{F}(x) =$