Journal of Computational Mathematics, Vol.23, No.2, 2005, 165–176.

FINITE VOLUME NUMERICAL ANALYSIS FOR PARABOLIC EQUATION WITH ROBIN BOUNDARY CONDITION *1)

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Abstract

In this paper, finite volume method on unstructured meshes is studied for a parabolic convection-diffusion problem on an open bounded set of R^d ($d = 2$ or 3) with Robin boundary condition. Upwinding approximations are adapted to treat both the convection term and Robin boundary condition. By directly getting start from the formulation of the finite volume scheme, numerical analysis is done. By using several discrete functional analysis techniques such as summation by parts, discrete norm inequality, et al, the stability and error estimates on the approximate solution are established, existence and uniqueness of the approximate solution and the 1st order temporal norm and L^2 and H^1 spacial norm convergence properties are obtained.

Mathematics subject classification: 65M12, 76M12.
Key words: Finite volume, Parabolic convection diffusion equations, Numerical analysis.

1. Introduction

Finite Volume Methods are known to be well applicable to the numerical simulation of many problems, particularly in the presence of convection terms, with irregular geometry domain or unstructed meshes partition. Many works have been done on their construction and application, as well as some theoretical studies [1]-[3]. Two main directions are usually followed to obtain their convergence properties. One is to write the finite volume as a finite element or mixed finite element method by some numerical integration, and follow the general finite element framework to prove the convergence (see, for instance, [3], where they come forward as generalized difference methods, and the citations of [1]). The second (see, for example, [1][2][4]) is to establish the convergence by using the direct formulation of the finite volume scheme together with appropriate discrete functional analysis techniques; following which, for elliptic equation, general boundary condition problems are studied in [1][2]; for parabolic equation, L^2 and H^1 error estimate only for Dirichlet boundary problem is considered respectively in [1][2] and [4].

In this paper, the finite volume discrete method on unstructured meshes including Voronoï or triangular meshes for parabolic convection-diffusion problem with a general Robin boundary condition is studied. The second approach, which is natural and direct to the original problem, is applied for numerical analysis. An "s" points (where s is the number of sides of each cell) finite volume scheme and an upstream scheme is adapted for the diffusion and the convection term respectively. An artificial upwinding is introduced in the treatment of the Robin boundary condition in order for the scheme to be well defined with no additional restriction on the mesh. But it brings difficulties and requires additional work for numerical analysis compared to that of the Dirichlet or Neumann case, which appears more evident for time involved parabolic

* Received July 7, 2003.
1) The project is supported by China National Key Program for Developing Basic (G1999032801), Mathematical Tianyuan Foundation (10226026), the National Natural Science Foundation of China (No.19932010) and the Foundation of CAEP (20040653).

problem compared with stationary elliptic problem. To solve this question, several discrete functional analysis techniques including summation-by-parts formula, discrete norm inequality, et al, are used. The stability and error estimates on the approximate solution are established. The existence and uniqueness of the approximate solution are shown. If the exact solution is at least in $L^\infty(0,T;H^2(\Omega))$, then the 1st order $L^\infty(0,T;L^2(\Omega))$ and $L^\infty(0,T;H^1(\Omega))$ norm convergence of the scheme are obtained.

Consider the parabolic problem with a Robin boundary condition:

$$\begin{aligned} u_t + \nabla\cdot(a\nabla u) + div(vu) + bu = f, &\quad x\in\Omega,\ t\in J,\\ a\nabla u\cdot\eta + \lambda u = g, &\quad x\in\partial\Omega,\ t\in J,\\ u(x,0) = u_0(x), &\quad x\in\Omega, \end{aligned} \tag{1.1}$$

where Ω is an open bounded subset of R^d ($d = 2$, or 3) which is a polygonal for $d = 2$ and polyhedral for $d = 3$ with $\partial\Omega$ its boundary, η is the unit normal to $\partial\Omega$ outward to Ω. $J = [0,T]$, with T a positive constant. $v = v(x,t)$ is a given vector function, $a = a(x,t), b = b(x,t), f = f(x,t), \lambda = \lambda(x,t), g = g(x,t)$ are given functions. Herein we study problem (1.1) with the following assumptions.

Assumption 1. For $t\in J$, $f(\cdot,t)\in L^2(\Omega)$, $b(\cdot,t)\in L^\infty(\Omega)$ and $v(\cdot,t)\in C^1(\bar\Omega)$ such that $div(v)/2 + b \geq 0$ almost everywhere (a.e.) on Ω.

Assumption 2. For $t\in J$, $g(\cdot,t)\in H^{\frac12}(\partial\Omega)$, $\lambda(\cdot,t)\in L^\infty(\partial\Omega)$ such that $v\cdot\eta/2+\lambda \geq 0\, a.e.\, on\, \partial\Omega$. Furthermore, if $v\cdot\eta/2+\lambda = 0\, a.e.\, on\, \partial\Omega$, then one assumes the existence of $\mathcal{O}\subset\bar\Omega$ such that its d-dimensional measure $m(\mathcal{O})\neq 0$ and such that $div(v)/2 + b \neq 0\, a.e.\, on\, \mathcal{O}$.

Assumption 3. For $t\in J$, $a(\cdot,t)$ is a piecewise C^1 function from $\bar\Omega$ to R, and there exists positive constant a_* such that $a(x,t)\geq a_*$ for a.e. $(x,t)\in\Omega\times J$.

Assumption 4. The functions a, b, λ, v and $div(v)$ are Lipschitz continuous with respect to t.

The outline of the paper is as follows. Section 2 introduces the restricted admissible meshes needed for the discretization, formulates the finite volume approximation and gives the definition of related spacial norms. Section 3 demonstrates corresponding numerical analysis, which includes the statement of stability and convergence properties and necessary reasoning procedure.

2. Finite Volume Discretization

2.1 Mesh Partition

Define the restricted admissible meshes as in [1], which includes meshes made with triangles and rectangles in two space dimensions, and Voronoï meshes.

Definition 1 (Restricted Admissible Meshes). *A finite volume mesh of Ω, denoted by $\mathcal{T}$, is given by a family of "control volumes", which are open polygonal (if $d = 2$) or polyhedral (if $d = 3$) convex subsets of Ω (with positive measure), a family of subsets of $\bar\Omega$ contained in hyperplanes of R^d, denoted by ε (these are the edges (if $d = 2$) or sides (if $d = 3$) of the control volumes), with strictly positive $(d-1)$-dimensional measure and a family of points of $\bar\Omega$ denoted by P. The finite volume mesh is called to be restricted admissible, if the properties (i) to (v) are satisfied.*

(i) The closure of the union of all the control volume is $\bar\Omega$.

(ii) For any $K\in\mathcal{T}$, there exists a subset ε_K of ε such that $\partial K = \bar K/K = \cup_{\sigma\in\varepsilon_K}\bar\sigma$. Let $\varepsilon = \cup_{K\in\mathcal{T}}\varepsilon_K$.

(iii) For any $(K,L)\in\mathcal{T}^2$ with $K\neq L$, either the $(d-1)$-dimensional Lebesgue measure of $\bar K\cap\bar L$ is 0 or $\bar K\cap\bar L = \bar\sigma$ for some $\sigma\in\varepsilon$, which will then be denoted by $K|L$.

(iv) The family $P = (x_K)_{K\in\mathcal{T}}$ is such that $x_K\in\bar K$ (for all $K\in\mathcal{T}$) and, if $\sigma = K|L$, it is assumed that $x_K\neq x_L$, and that the straight line $D_{K,L}$ going through x_K and x_L is orthogonal to $K|L$.