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INTERPOLATION BY LOOP'S SUBDIVISION FUNCTIONS *1)

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Abstract

For the problem of constructing smooth functions over arbitrary surfaces from discrete data, we propose to use Loop's subdivision functions as the interpolants. Results on the existence, uniqueness and error bound of the interpolants are established. An efficient progressive computation algorithm for the interpolants is also presented.

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1. Introduction

The problem of constructing interpolants on surfaces arises in some application areas such as characterizing the rain fall on the earth, the pressure on the wing of an airplane and the temperature on a human body. The problem was first proposed as an open question by Barnhill [5] in 1985. After that, a considerable number methods have been developed for dealing with it (for surveys see [7], [12]). Most of these methods interpolate the scattered data over planar or spherical domain surfaces. In [8] and [11], the domains are generalized to convex surface and topological genus zero surface, respectively. Pottmann [17] presented a method which does not possess similar restrictions on the domain surface but requires it to be of C^2. In [6] this restriction was left and the function on surface is constructed by transfinite interpolation. It seems that, the currently known approaches possess restrictions either on domain surfaces or functions on surfaces. The domain surfaces are usually assumed to be spherical, convex or genus zero. The functions on surfaces are not always polynomial [6], [15] or rather higher order polynomial [18]. The aim of this paper is to design a low order piecewise polynomial interpolation scheme over triangulated surfaces.

In several recent developments in computer graphics and numerical analysis (see [2, 3, 4, 9, 10]), Loop's subdivision (see [14]) surfaces and functions on surfaces have played a key role. In these developments, Loop's subdivision surfaces and function on surfaces are used to construct the finite element function space in a discretization process of a partial differential equation. However, the convergence analysis or error estimation in these discretization process require the interpolation error estimation by the function in the finite element function space. Such a result currently is not available. In this paper, we estimate the interpolation error bound and further provide an efficient method for constructing smooth multi-resolution functions over a surface. Precisely, we consider the following problem:

Given a discretized triangular surface mesh $T \subset I\!R^3$ and a discretized function $D \subset I\!R^\kappa$. Each of the function values is attached to one vertex of the surface mesh. Our primary goal is to construct smooth (non-discretized) representations for the surface functions that interpolate the discretized data. Our secondary goal is to estimate the error of the interpolation. Our tertiary goal is to establish a progressive computational method for the interpolation functions.

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We propose to use Loop's subdivision functions as the interpolants. Results on the existence, uniqueness and error bound of the interpolants are established. An efficient progressive computation algorithm for the interpolants is also presented.

The rest of the paper is organized as follows. In Section 2 we review some basic aspects on Loop's subdivision. In Section 3, we formulate the interpolation problem and then establish the result on the solvability of the interpolation problem. Section 4 is devoted to the interpolation error and convergence and Section 5 is for the efficient computation of the interpolation functions. Numerical examples are given in Section 6.

2. Loop's Subdivision Surfaces and Functions

Let us introduce some notations used in this paper:

S : domain surface of the interpolation, the limit surface of Loop's subdivision.

T : a triangulation of S.

$T^{(k)}$: a sequence of triangulation of S.

M : control mesh of T.

$M^{(k)}$: control mesh of $T^{(k)}$.

In Loop's subdivision scheme, the initial control mesh $M^{(0)}$ and the subsequent refined meshes $M^{(k)}$ consist of triangles only. In the refinement, each triangle is subdivided into 4 sub-triangles. Then the vertex position of the refined mesh is computed as the weighted average of the vertex position of the unrefined mesh. Consider a vertex x_0^k at level k with neighbor vertices x_i^k for $i = 1, \cdots, n$, where n is the valence of vertex x_0^k. The positions of the newly generated vertices x_i^{k+1} on the edges of the previous mesh are computed as

$$x_i^{k+1} = \frac{3x_0^k + 3x_i^k + x_{i-1}^k + x_{i+1}^k}{8}, \quad i = 1, \cdots, n, \tag{2.1}$$

where index i is to be understood modulo n. The old vertices get new positions according to

$$x_0^{k+1} = (1 - na)x_0^k + a\left(x_1^k + x_2^k + \cdots + x_n^k\right), \tag{2.2}$$

where $a = \frac{1}{n}\left[\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}cos\frac{2\pi}{n}\right)^2\right]$. Note that all newly generated vertices have a valence of 6, while the vertices inherited from the original mesh at level zero may have a valence other than 6. We will refer to the former case as *ordinary* and to the later case as *extraordinary*. The limit surface S of Loop's subdivision is C^2 everywhere except at the extraordinary points where it is C^1.

2.1. The Limit Surface Corresponding to Vertices

Lemma 2.1. *Let x_0^0 be a vertex with x_i^0, $i = 1, \cdots, n$, being the 1-ring neighbor vertices of the initial control mesh $M^{(0)}$. Then all these vertices converge to a single position*

$$v_0^T := (1 - nl)x_0^0 + l\sum_{i=1}^{n} x_i^0, \quad l = 1/[n + 3/(8a)] \tag{2.3}$$

as the subdivision step goes to infinity (see [4] for the proof of the Lemma).

Let x_0^1, x_i^1, $i = 1, \cdots, n$ be the control vertices generated by subdivision once around x_0^0 of the initial control mesh $M^{(0)}$. Then

$$v_0^T = (1 - nl)x_0^1 + l\sum_{i=1}^{n} x_i^1. \tag{2.4}$$