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SYMMETRIC INTERIOR PENALTY DG METHODS FOR THE COMPRESSIBLE NAVIER–STOKES EQUATIONS I: METHOD FORMULATION

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Abstract. In this article we consider the development of discontinuous Galerkin finite element methods for the numerical approximation of the compressible Navier–Stokes equations. For the discretization of the leading order terms, we propose employing the generalization of the symmetric version of the interior penalty method, originally developed for the numerical approximation of linear self-adjoint second–order elliptic partial differential equations. In order to solve the resulting system of nonlinear equations, we exploit a (damped) Newton–GMRES algorithm. Numerical experiments demonstrating the practical performance of the proposed discontinuous Galerkin method with higher–order polynomials are presented.

Key Words. Discontinuous Galerkin methods, a posteriori error estimation, adaptivity, compressible Navier–Stokes equations

1. Introduction

In recent years there has been tremendous interest in the design of discontinuous Galerkin finite element methods (DGFEMs, for short) for the discretization of compressible fluid flow problems; see, for example, [3, 4, 5, 8, 9, 10, 11, 24] and the references cited therein. The key advantages of these schemes are that DGFEMs provide robust and high-order accurate approximations, particularly in transport-dominated regimes, and that they are locally conservative. Moreover, there is considerable flexibility in the choice of the mesh design; indeed, DGFEMs can easily handle non-matching grids and non-uniform, even anisotropic, polynomial approximation degrees. Additionally, orthogonal bases can easily be constructed which lead to diagonal mass matrices; this is particularly advantageous for unsteady problems. Finally, in combination with block-type preconditioners, DGFEMs can easily be parallelized.

In our previous work, see the series of papers [16, 17, 18, 21], for example, we have been concerned with the development of DGFEMs for the numerical approximation of inviscid compressible fluid flows, coupled with automatic adaptive mesh generation. In particular, the key focus of these articles was the derivation of so–called 'goal–oriented' *a posteriori* error bounds together with the design of corresponding adaptive mesh refinement algorithms in order to yield guaranteed error control; for the generalization of these ideas to the *hp*–version of the DGFEM, we refer to the article [28] and the references cited therein. In contrast to traditional *a posteriori* error estimation which seeks to bound the error with respect to a given norm,

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goal–oriented *a posteriori* error estimation bounds the error measured in terms of certain target functionals of real or physical interest. Typical examples include the mean value of the field over the computational domain Ω, the normal flux through the outflow boundary of Ω, the evaluation of the solution at a given point in Ω and the drag and lift coefficients of a body immersed in a fluid. For related work, we refer to [6, 24], for example.

The purpose of this article and its companion–article [20] is to extend our earlier work on DGFEMs for nonlinear systems of first–order hyperbolic conservation laws to the compressible Navier–Stokes equations. For the discretization of the leading order terms, we propose employing the symmetric version of the interior penalty DGFEM. One of the key aspects of this discretization scheme is the satisfaction of the adjoint consistency condition, cf. [1], for linear problems. This condition is essential to guarantee that the optimal order of convergence of the numerical approximation to the underlying analytical solution is attained when the discretization error is measured in terms of either the L_2–norm, or in the 'goal–oriented' setting, in terms of a given target functional of practical interest. This property is *not* shared by, for example, the non-symmetric version of the interior penalty DGFEM, cf. [14]. Indeed, this latter method is sub-optimal by a full order of the mesh size h, when the error is measured in terms of the L_2-norm, for even polynomial degrees; though it has been noted experimentally, that the optimal rate of convergence of the scheme is achieved for odd orders, cf. [22, 25].

The paper is structured as follows. After introducing, in Section 2, the compressible Navier–Stokes equations, in Section 3 we formulate its discontinuous Galerkin finite element approximation; here, a detailed description of the implementation of the corresponding boundary conditions is also outlined. Section 4 is devoted to the practical implementation of the underlying discretization method; in particular, here we propose a damped Newton–GMRES algorithm for the solution of the system of nonlinear equations arising from the DGFEM discretization of the underlying PDE system. In Section 5 we present a series of numerical experiments to illustrate the performance of the proposed symmetric interior penalty DGFEM when higher–order polynomial degrees are employed. In particular, we demonstrate the performance of the nonlinear Newton iteration with different preconditioning strategies. Then, we compare the convergence of force coefficients under both global and local grid refinement for a standard laminar test case, as well as highlighting the numerical resolution of boundary layer profiles when linear and higher-order polynomial degrees are employed. Finally, in Section 6 we summarize the work presented in this paper and draw some conclusions.

2. The compressible Navier-Stokes equations

We consider the two–dimensional steady state compressible Navier-Stokes equations. Writing ρ, $\mathbf{v} = (v_1, v_2)^T$, p, E and T to denote the density, velocity vector, pressure, specific total energy and temperature, respectively, the equations of motion are given by

$$\nabla \cdot (\mathcal{F}^c(\mathbf{u}) - \mathcal{F}^v(\mathbf{u}, \nabla\mathbf{u})) \equiv \frac{\partial}{\partial x_i}\mathbf{f}_i^c(\mathbf{u}) - \frac{\partial}{\partial x_i}\mathbf{f}_i^v(\mathbf{u}, \nabla\mathbf{u}) = 0 \quad \text{in } \Omega, \tag{1}$$

where Ω is an open bounded domain in $\mathbb{R}^2$; here, and throughout the rest of this article, we use the summation convention, i.e., repeated indices are summed through their range. The vector of conservative variables $\mathbf{u}$ and the convective fluxes $\mathbf{f}_i^c$,