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CONVERGENCE AND STABILITY OF IMPLICIT METHODS FOR JUMP-DIFFUSION SYSTEMS

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Abstract. A class of implicit methods is introduced for Ito stochastic differential equations with Poisson-driven jumps. A convergence proof shows that these implicit methods share the same finite time strong convergence rate as the explicit Euler–Maruyama scheme. A mean-square linear stability analysis shows that implicitness offers benefits, and a natural analogue of mean-square A-stability is studied. Weak variants are also considered and their stability analyzed.

Key Words. A-stability, backward Euler, Euler–Maruyama, linear stability, Poisson process, stochastic differential equation, strong convergence, theta method, trapezoidal rule.

1. Introduction

Applications in economics, finance, and several areas of science and engineering, give rise to jump-diffusion Ito stochastic differential equations [2, 4, 24] of the form

$$dX(t) = f(X(t^-))dt + g(X(t^-))dW(t) + h(X(t^-))dN(t), \quad t > 0, \tag{1}$$

with $X(0)$ given, where $X(t^-)$ denotes $\lim_{s\to t^-} X(s)$. Here, $f : \mathbb{R}^n \to \mathbb{R}^n$ is the drift coefficient, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is the diffusion coefficient and $W(t)$ is an m-dimensional Brownian motion. We assume that $N(t)$ is a scalar Poisson process with intensity λ, and hence the jump coefficient has the form $h : \mathbb{R}^n \to \mathbb{R}^n$. Extension of our work to vector-valued jumps with independent entries is straightforward. Conditions on the coefficients and initial data that guarantee a unique solution will be introduced in section 2.

We consider a class of theta methods for (1). For a constant stepsize $\Delta t > 0$ and a particular choice of $\theta \in [0, 1]$, the theta method is defined by $Y_0 = X(0)$ and

$$Y_{n+1} = Y_n + (1-\theta)f(Y_n)\Delta t + \theta f(Y_{n+1})\Delta t + g(Y_n)\Delta W_n + h(Y_n)\Delta N_n. \tag{2}$$

Here, Y_n is the approximation to $X(t_n)$, for $t_n = n\Delta t$, with $\Delta W_n := W(t_{n+1}) - W(t_n)$ and $\Delta N_n := N(t_{n+1}) - N(t_n)$ denoting the increments of the Brownian motion and the Poisson process, respectively.

We refer to (2) as a class of theta methods because in the deterministic ordinary differential equation (ODE) case, where $g(\cdot) \equiv h(\cdot) \equiv 0$ and $X(0)$ is constant, (2) reduces to the well-known class with this name. For Ito stochastic differential equations (SDEs), where $h(\cdot) \equiv 0$, the class has been referred to as the semi-implicit Euler method [11, 23] and the stochastic theta method [9]. Our motivation

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for introducing and studying (2) is that for the ODE and SDE cases, it has been found that the class offers good linear stability properties [8, 9, 23] and excellent potential for capturing long time dynamics [20, 25]. Our aim here is to show that the theta method offers a means to define useful implicit integrators in the presence of jumps. Section 2 justifies the methodology by giving a finite time strong convergence proof. Section 3 analyzes mean-square stability and quantifies precisely what may be gained by moving away from the Euler–Maruyama ($\theta = 0$) case. Stability for a weak version of the theta method is studied in section 4.

Previous work on numerical methods for jump-diffusion problems includes [3, 6, 7, 12, 14, 15, 16, 17, 21, 22]. The references [6, 7, 12, 14, 21, 22] deal with weak convergence. In [6, 7, 12, 21] 'jump-adapted' explicit methods that directly incorporate the jump points are studied, whereas [14, 22] use a fixed Δt. Glasserman [5, page 364] points out that jump-adaption may be expensive when the jump intensity λ is large. Strong convergence for fixed stepsize explicit methods is studied in [3, 15, 16, 17]. Our work differs from these references in that (a) implicit methods are considered, and (b) in addition to finite time strong convergence, mean-square stability properties are analyzed.

2. Strong Convergence Proof

In this section we suppose that the problem (1) is to be solved over a finite time interval, $[0, T]$, where T is a constant. We study classical strong convergence, and hence we are concerned with the regime where $\Delta t \to 0$ with T fixed. The reference [1] mentions a number of applications where this type of convergence is required, the most relevant for our work being mathematical finance. The initial steps of the proof follow the ideas in [10, Appendix A], where a strong convergence result for the theta method on a non-jump SDE is given. Our proof is more general in that it deals with the jump term and also places the supremum over time inside the expectation operator (see Theorem 2.4 below).

Letting $|\cdot|$ denote both the Euclidean vector norm and the Frobenius matrix norm, we assume that f, g, h satisfy the global Lipschitz condition:

$$|a(x) - a(y)|^2 \leq K|x - y|^2, \qquad \text{for } a \equiv f, g, \text{ or } h, \tag{3}$$

where K is a constant independent of x and y, and we note that this implies the linear growth bound

$$|a(x)|^2 \leq L(1 + |x|^2), \qquad \text{for } a \equiv f, g, \text{ or } h, \tag{4}$$

where L is a constant independent of x and y. Our assumption on the initial data is that $\mathbb{E}|X(0)|^2$ is finite and $X(0)$ is independent of $W(t)$ and $N(t)$ for all $t \geq 0$. We note that these conditions imply the existence of a unique solution for (1), see, for example, [4, 24].

For convenience, we will extend the discrete numerical solution to continuous time. We first define the 'step functions'

$$Z_1(t) = \sum_k Y_k \mathbf{1}_{[k\Delta t,(k+1)\Delta t)}(t), \quad Z_2(t) = \sum_k Y_{k+1} \mathbf{1}_{[k\Delta t,(k+1)\Delta t)}(t), \tag{5}$$

where $\mathbf{1}_G$ is the indicator function for the set G. Then we define

$$\begin{aligned} Y(t) \;&=\; Y_0 + \int_0^t (1-\theta) f(Z_1(s))\, ds + \int_0^t \theta f(Z_2(s))\, ds + \int_0^t g(Z_1(s))\, dW(s) \\ &\quad + \int_0^t h(Z_1(s))\, dN(s). \end{aligned} \tag{6}$$