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NUMERICAL METHODS FOR THE EXTENDED FISHER-KOLMOGOROV (EFK) EQUATION

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Abstract. In the study of pattern formation in bi–stable systems, the extended Fisher–Kolmogorov (EFK) equation plays an important role. In this paper, some *a priori* bounds are proved using Lyapunov functional. Further, existence, uniqueness and regularity results for the weak solutions are derived. Using C^1-conforming finite element method, optimal error estimates are established for the semidiscrete case. Finally, fully discrete schemes like backward Euler, two step backward difference and Crank-Nicolson methods are proposed, related optimal error estimates are derived and some computational experiments are discussed.

Key Words. Extended Fisher-Kolmogorov (EFK) equation, Lyapunov functional, weak solution, existence, uniqueness and regularity results, finite element method, semidiscrete method, backward Euler, two step backward difference and Crank-Nicolson schemes, optimal estimates.

1. Introduction

In this paper, the C^1-conforming finite element method is analyzed for the following extended Fisher-Kolmogorov (EFK) equation :

$$u_t + \gamma \Delta^2 u - \Delta u + f(u) = 0, \quad (x,t) \in \Omega \times (0,T], \tag{1.1}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

either of the boundary conditions

$$u = 0, \frac{\partial u}{\partial \nu} = 0, \quad (x,t) \in \partial\Omega \times (0,T], \tag{1.3}$$

or

$$u = 0, \ \Delta u = 0, \ (x,t) \in \partial \times (0,T], \tag{1.4}$$

where $f(u) = u^3 - u$, $T > 0$ and Ω is a bounded domain in $\Re^d$, $d \leq 2$ with boundary $\partial\Omega$.

When $\gamma = 0$ in (1.1), we obtain the standard Fisher-Kolmogorov equation. However, by adding a stabilizing fourth order derivative term to the Fisher-Kolmogorov equation, Coullet *et al.* [4], Dee and van Saarloos [7, 19, 20] proposed (1.1) and called the model described in (1.1) as the extended Fisher-Kolmogorov equation.

The equation (1.1) occurs in a variety of applications such as pattern formation in bi-stable systems [7], propagation of domain walls in liquid crystals [22], travelling waves in reaction diffusion systems [2] and mezoscopic model of a phase transition in a binary system near the Lipschitz point [8]. In particular, in the

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phase transitions near critical points (Lipschitz points), the higher order gradient terms in the free energy functional can no longer be neglected and the fourth order derivative becomes important.

Recently, attention has been focused on the steady state equation of (1.1). The aim of considering the steady state equation of (1.1) is to study the heteroclinic solutions (so called kinks) connecting to the equilibria $u = -1$ and $u = 1$. Typically, the stationary problem displays a multitude of periodic, homoclinic and heteroclinic solutions [13, 15] depending on the parameter γ. The steady state equation of (1.1) has been analysed by Peletier and Troy [13, 14] using shooting methods and by Kalies, Kwapisz and Vander Vorst [9] with the help of variational methods.

As far as computational studies are concerned, there is hardly any literature for the numerical approximations to (1.1)–(1.3) or (1.1)–(1.2) and (1.4). Therefore, an attempt has been made here to discuss finite element Galerkin method for the EFK equation. In this article, we mainly concentrate on the equation (1.1) with the initial condition (1.2) and boundary conditions (1.3). Related to fourth order evolution equations, the C^1-conforming finite element method is analyzed by Pani and Chung [11] for the Rosenau equation, for "Good" Boussinesq equation by Pani and Haritha [12], for one dimensional Cahn-Hilliard equation by Elliott *et., al* [5, 6], for multidimensional Cahn-Hilliard equation by Qiang and Nicolaides [16] and for Kuramoto-Sivashinsky equation by Akrivis [1].

The outline of the paper is as follows. Section 2 deals with existence, uniqueness and regularity results. In section 3, we derive *a priori* error estimates for the semidiscrete Galerkin method using C^1-conforming finite elements. In section 4, we discretize the semidiscrete equation in the temporal direction and obtain optimal error estimates for the backward Euler, two step backward difference and Crank-Nicolson schemes. Finally in section 5, we discuss some computational experiments.

2. Existence, Uniqueness and Regularity results

In this section, we derive existence uniqueness and regularity results for the extended Fisher-Kolmogorov (EFK) equation. In literature, we observe that there is hardly any study on the existence, uniqueness and regularity results of weak solutions to the problem (1.1)–(1.3) or (1.1)–(1.2) and (1.4). Therefore, an attempt has been made in this section to derive existence, uniqueness and regularity results for the EFK equation (1.1)–(1.3).

Taking L^2-innerproduct of (1.1) with $\chi \in H_0^2$ and applying Green's formula, we obtain the following weak formulation. Find $u(\cdot, t) \in H_0^2$ for $t \in (0, T]$ such that

$$\begin{aligned} (u_t, \chi) + \gamma(\Delta u, \Delta\chi) + (\nabla u, \nabla\chi) \quad &+ \quad (f(u), \chi) = 0, \quad \chi \in H_0^2(\Omega), \\ u(0) \quad &= \quad u_0. \end{aligned} \tag{2.1}$$

For the proof of existence and uniqueness results, the following *a priori* bound will be useful.

Theorem 2.1. *Assume that $u_0 \in H_0^2$. Then there exists a positive constant C such that*

$$\|u(t)\|_2 \leq C(\gamma, \|u_0\|_2), \quad t > 0.$$

Further,

$$\|u(t)\|_\infty \leq C(\gamma, \|u_0\|_2), \quad t > 0.$$

Proof. We consider the Lyapunov functional $\mathcal{E}(\chi)$ as

$$\mathcal{E}(\chi) = \int_\Omega \{\frac{\gamma}{2}|\Delta\chi|^2 + \frac{1}{2}|\nabla\chi|^2 + F(\chi)\}\, dx, \tag{2.2}$$