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ON THE CENTRAL RELAXING SCHEME II: SYSTEMS OF HYPERBOLIC CONSERVATION LAWS*1)

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Abstract

This paper continues to study the central relaxing schemes for system of hyperbolic conservation laws, based on the local relaxation approximation. Two classes of relaxing systems with stiff source term are introduced to approximate system of conservation laws in curvilinear coordinates. Based on them, the semi–implicit relaxing schemes are constructed as in [6, 12] without using any linear or nonlinear Riemann solvers. Numerical experiments for one–dimensional and two–dimensional problems are presented to demonstrate the performance and resolution of the current schemes.

Key words: Hyperbolic conservation laws, The relaxing system, The central relaxing schemes, The Euler equations.

1. Introduction

We are interested in construction of the central relaxing schemes for system of nonlinear hyperbolic conservation laws

$$\frac{\partial \mathbf{U}}{\partial t} + \sum_{i=1}^{d} \frac{\partial \mathbf{F}_i(\mathbf{U})}{\partial x_i} = 0, \tag{1.1}$$

with initial data $\mathbf{U}(0, \mathbf{x}) = \mathbf{U}_0(\mathbf{x})$, $\mathbf{x} = (x_1, ..., x_d)$, based on the local relaxation approximation of Eq.(1.1) [2, 3, 6, 8, 9, 12].

To illustrate the basic idea of the relaxing schemes, for the sake of simplicity in the presentation, we restrict our attention to one–dimensional scalar conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0. \tag{1.2}$$

First, introduce a linear hyperbolic system with a stiff source term (hereafter called the *relaxing system*) :

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} + a\frac{\partial u}{\partial x} &= -\frac{1}{\epsilon}(v - f(u)), \end{aligned} \tag{1.3}$$

to approximate (1.2), where the small positive parameter ϵ is the relaxation rate, and a is a positive constant satisfying

$$| f'(u) | \leq \sqrt{a}, \quad \forall u \in R. \tag{1.4}$$

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In the small relaxation limit $\epsilon \to 0^+$, the relaxing system (1.3) can be approximated to leading order by the following *relaxed* equations

$$v = f(u), \tag{1.5}$$
$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0. \tag{1.6}$$

The state satisfying (1.5) is usually called the *local equilibrium*. By the Chapman-Enskog expansion [1], we can derive the following first order approximation to (1.3)

$$v = f(u) - \epsilon\{a - [f'(u)]^2\}\frac{\partial u}{\partial x}, \tag{1.7}$$
$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \epsilon\frac{\partial}{\partial x}(\{a - [f'(u)]^2\}\frac{\partial u}{\partial x}). \tag{1.8}$$

It is clear that the above second equation (1.8) is a dissipative approximation to (1.2) under condition (1.4) (which is also referred to as the *subcharacteristic condition* [8]).

The second step is to discrete the relaxing system (1.3) in a proper way (see [6, 7, 12, 13] for details). To avoid the initial layer introduced by the relaxing system (1.3), we can choose the special initial condition for the relaxing system (1.3):

$$\begin{cases} u(x,0) = u_0(x), \\ v(x,0) = v_0(x) \equiv f(u_0(x)). \end{cases} \tag{1.9}$$

In doing so the state is already in equilibrium initially. On the other hand, to avoid any new boundary layers in solving boundary value problems, we can also impose the boundary conditions for v that are consistent to the local equilibrium.

The relaxation limit for systems of conservation laws with a stiff source term was first studied by Liu in [8]. Convergence of solutions of the general relaxing systems are considered later in [2, 3, 9]. Jin and Xin [6] first considered numerical approximations of conservation laws by using the *relaxing systems* and presented a class of nonoscillatory upwind relaxing schemes. Tang and Wu [13] have analyzed a cell entropy inequality of their upwind relaxing schemes. The main advantage of numerically solving the *relaxing systems* (1.3) over the original equation (1.1) lies in its special structures of the linear characteristic fields and localized source term. Numerically solving the *relaxing system* (1.3) enables one to avoid nonlinear Riemann solvers spatially.

Numerical schemes for stiff relaxing systems such as (1.3) were studied in [7]. Proper implicit time discretizations should be taken to overcome the stability constraints brought by the stiff source. Since the source terms in form (1.3) is linear in the variable v, a simple way is to keep the convection terms explicit and the stiff source terms implicit.

However, the numerical experiments have shown that the implementation of their upwind relaxing schemes for general hyperbolic system is more relatively complicated, because of using linear Riemann solvers of a new hyperbolic system with a stiff source term spatially and the choice of a for different problem. Moreover, its cost is too much.

To overcome these drawback and simplify the costly characteristic procedure, we have constructed a class of the central relaxing schemes for scalar conservation laws in [12] without using linear or nonlinear Riemann solvers. The schemes are shown to be TVD (total variation diminishing) and be of the similar relaxed form as one in [6] in the zero relaxation limit For scalar equations, a cell entropy inequality for semidiscrete schemes has also been studied.

This paper continues to study central relaxing schemes for systems of conservation laws, based on using the local relaxation approximation. Two relaxing systems with stiff source term will be introduced for hyperbolic conservation laws in curvilinear coordinates. Numerical experiments for 1D and 2D problems are also presented to demonstrate the performance and resolution of our central relaxing scheme, and in comparison with the upwind relaxing schemes.