INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 2, Number 1, Pages 31–42

ORTHOGONALITY CORRECTION TECHNIQUE IN SUPERCONVERGENCE ANALYSIS

CHUANMIAO CHEN

(Communicated by Zhimin Zhang)

Abstract. A technique of orthogonality correction in an element is introduced and applied to superconvergence analysis in finite element method. Ultraconvergence results for rectangular elements of odd degree $n \geq 3$ are derived in the case of variable coefficients.

Key Words. Finite elements, rectangular element, orthogonality correction, superconvergence and ultraconvergence.

1. Introduction

Consider a second order elliptic problem with Dirichlet condition (BV1)

$$Au \equiv -D_j(a_{ij}D_iu) + a_0u = f \text{ in } \Omega,\ u = 0 \text{ on } \Gamma, \tag{1}$$

where Ω is a planar polygonal domain with the boundary Γ. Denote by $W^{k,p}(\Omega)$ Sobolev space with norm

$$||u||_{k,p,\Omega} = (\int_\Omega \sum_{|\alpha|\leq k} |D^\alpha u(x)|^p dx)^{1/p}.$$

If $p = 2$, the subscript p is often omitted and we simply use $H^k(\Omega)$ and $||u||_{k,\Omega}$. Assume that the domain Ω is subdivided into a finite number of elements τ (with h the largest diameter of all τ) and its mesh J^h is quasiuniform.

Introduce the following subspace

$$V = \{u:\ u \in H^1(\Omega), u = 0 \text{ on } \Gamma\}$$

and the n-degree finite element subspace by

$$S^h = \{v:\ v \in C(\Omega),\ v|_{\tau_j} \in P_n,\quad \tau_j \in J^h, v|_\Gamma = 0\}.$$

Define the bilinear form and inner product

$$A(u,v) = \int_\Omega (a_{ij}D_iuD_jv + a_0uv)dx,\ f(v) = (f,v)$$

and assume that $A(u,u)$ is V-coercive. We know that the true solution $u \in V$ and its finite element approximation $u_h \in S^h$ satisfy the following orthogonal relation

$$A(u-u_h, v) = 0,\quad v \in S^h. \tag{2}$$

It is well known that under some conditions there are some basic error estimates

$$\|u-u_h\|_{j,\Omega} \leq ch^{n+1-j}\|u\|_{n+1,\Omega},\quad j = 0,1;\ n \geq 1. \tag{3}$$

Received by the editors June 16, 2004 and, in revised form, October 13, 2004.
2000 *Mathematics Subject Classification.* 65N30.

and negative norm estimates (if $n \geq 2,\ 2 \leq s \leq n+1$)

$$(4) \qquad \|u-u_h\|_{-l,\Omega} = \sup_{v \in H^l(\Omega)} \frac{|(u-u_h, v)|}{||v||_l} \leq ch^{s+l}\|u\|_{s,\Omega},\ 1 \leq l \leq n-1,$$

in which $||u-u_h||_{1-n} = O(h^{2n})$ is of superconvergence of the highest order known.

(4) seems to imply two kinds of ideas to study superconvergence. First, (4) means the approximate orthogonality of $e = u - u_h$, because of the arbitrariness of v, which is a local property (generally, the approximate orthogonality is invalid in an element) and leads to various local averaging. Second, (4) also means that (e, v) is of higher accuracy than $||e||$. As a result, the error $e = u - u_h$ has to change rapidly its sign in Ω in order that the cancellation of the positive and negative values makes the integral less. We want to know whether the distribution of zeroes of e has certain regular patterns, and whether it is possible to find its approximate points independent of the coefficients of A and the concrete behavior of u. This is the study of superconvergence points.

Up to now, there are five main methods:

1. The local averaging method. Motivated by the negative norm estimates and interior estimates, applying the splines on a uniform mesh to construct the kernel function K_h^α with small support and α-order difference quotient $\partial_h^\alpha u_h$, Bramble and Schatz [2] (1974-77) and Thomee [28](1977) obtained the high accurate convolution

$$K_h^\alpha * \partial_h^\alpha u_h - D^\alpha u = O(h^{2n}),\ \text{in}\ \Omega_0 \subset\subset \Omega.$$

Here both function and derivatives of any order are of optimal order superconvergence $O(h^{2n})$. This is incompatible for other methods. Later these results were extended to parabolic equation (Thomee [29] 1980) and nonlinear problems (Chen [6] 1983).

2. Quasi-projection and Tensor Product Method. It is adopted by Douglas, Dupont and Wheeler [17](1974). This method requires the coefficient $a_{12} = 0$ and the use of tensor product polynomials. The tensor product idea is clear and powerful. We remark that it can be applicable to the time-space full discrete problems.

3. Element Orthogonality Analysis (EOA). It was started by Zlamal [39][40](1977-78), and independently found by Chen [3][4][5] (1978-81) at el. Many other scholars worked in this aspect, such as Zhu[36], Lin-Xu [23],Chen and Huang [13], Krizek- Neittaamäki [19], et al. EOA is based on orthogonal approximations in the bilinear inner product sense and it doesn't depend directly on the estimates for PDEs. Therefore, this method is applicable to more general equations, and the corresponding conclusions are often valid up to the boundary (mainly under BV1). To solve the general equations, Chen proposed three important techniques: cancellation technique between elements, orthogonal expansion and orthogonality correction in an element. Chinese scholars have finished the systematical work in this approach, see Chen [12], Chen-Huang [13] and Lin-Yan [24] and so on.

4. Computer-based research. Babuska-Strouboulis et al. in 1995 (see [1]) finished a systematical computational search for superconvergence points of derivatives and drew a lot of valuable conclusions. In particular, a surprising structure of superconvergence for triangular elements of degree $1 \sim 7$ is first exhibited. Their research has shown a very promising future for new approaches.