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PETROV-GALERKIN METHOD WITH LOCAL GREEN'S FUNCTIONS IN SINGULARLY PERTURBED CONVECTION-DIFFUSION PROBLEMS

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Abstract. Previous theoretical and computational investigations have shown high efficiency of the local Green's function method for the numerical solution of singularly perturbed problems with sharp boundary layers. However, in several space variables those functions, used as projectors in the Petrov-Galerkin scheme, cannot be derived in a closed analytical form. This is an obstacle for the application of the method when applied to multi-dimensional problems. The present work proposes a semi-analytical approach to calculate the local Green's function, which opens a way to effective practical application of the method. Besides very accurate approximation, the matrix stencils obtained with these functions allow the use of fast and stable iterative solution of the large sparse algebraic systems that arise from the grid-discretization. The advantages of the method are illustrated by numerical examples.

Key Words. Convection-diffusion equation, Petrov-Galerkin discretization, Fourier transform, integral equations, iterative solution.

1. Introduction

Singularly perturbed problems are generally acknowledged to be a hard task for numerical evaluation. Due to their solutions having sharp boundary and interior layers severe numerical instability can occur and a large error pollution spreads out over the whole domain as the perturbation parameter tends to its limit value.

A classical example of a singularly perturbed equation is the equation of the convection-diffusion problem:

$$\mathcal{L}u \equiv -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu = f, \qquad (x,y)\in\Omega\subset R^2 \tag{1.1}$$

where ε is a small parameter and $c \geq 0$. Here we denote scalar products of algebraic vectors by dots, as in the convection term $\mathbf{b}\cdot\nabla u$, while the brackets notation is reserved for the scalar products in the functional Hilbert spaces occurring below.

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For the sake of definiteness let us consider a rectangular domain $\Omega:\ 0 \le x \le a_1,\ 0 \le y \le a_2$ with homogeneous Dirichlet boundary condition

$$u|_\Gamma = 0, \qquad \Gamma = \partial\Omega. \tag{1.2}$$

Only the presence of the diffusion term $-\varepsilon\Delta u$ enables fulfillment of this condition at the outflow part Γ_- of the boundary Γ entailing a boundary layer of width $O(\varepsilon)$ near Γ_- (here $\Gamma_- = \{(x,y) \in \Gamma:\ \mathbf{b}\cdot\mathbf{n} > 0\}$, $\mathbf{n}$ is an outward normal to $\partial\Omega$).

It is well known that if the standard Galerkin method, or the similar central difference method, is used in regions where layers occur, then unphysical oscillations arise. They can be suppressed by the use of a locally and significantly refined grid where a certain local Peclét number condition is satisfied [3]. Instead of the standard Galerkin method, the streamline upwind method (see [15]) is a popular method to stabilize the scheme. One more alternative to the Galerkin scheme, applicable to singularly perturbed problems, is the meshless local Petrov-Galerkin method [2].

Other methods used are related to the local characteristic line method, which is based on the property that away from the layers the solution follows narrowly the characteristic lines for the reduced equation ($\varepsilon = 0$), when ε is small. Using such methods with a proper locally refined grid, under certain assumptions regarding the influence of corner singularities, one can prove optimal order discretization error estimates, typically of $O(h^2)$, which hold uniformly in the singular perturbation parameter, see [4, 8, 18]. Other and related papers can be found in [16].

Among the variety of approaches to regularization of singularly perturbed problems there is a group of methods based on the use of Green's functions associated with the equations considered. Different kinds of Green's functions are used for preconditioning [12] or for construction the fine-scale spaces in the multiscale stabilized methods [9]. The present paper deals with the method [5], which may be considered as a special case of the approach [2] with the local Green's functions used as projectors (test space). In distinction to the global Green's functions used in [12], the local ones are set on elementary supports defined by discretization. On the other hand, unlike to the fine-scale Green's functions of the multiscale decomposition methods, they are connected with a coarse mesh and do not possess, therefore, the property to provide an hierarchical exact basis of the infinitely dimension fine-scale spaces. Nevertheless, the local Green's functions also provide the stabilization effect by accounting *indirectly* for the fine-scale behavior. In this regard the method proposed may be treated as a specific implementation of the multiscale decomposition theory [10].

Let us give the main idea of the presented approach with a simple example of a uniform grid approximation. Let

$$u_h(x,y) = \sum_{k=1}^{N} u_k\varphi_k(x,y) \tag{1.3}$$

be an expansion of the exact solution of the problem (1.1) and (1.2) in terms of basis functions $\varphi_k = \varphi((x-x_k)/h, (y-y_k)/h)$ defined at the interior nodes (x_k, y_k) of a grid covering Ω with a step h (Fig. 1); $\varphi(x,y)$ is a shape-function. In line with the Petrov-Galerkin scheme, the unknown coefficients u_k are determined from the