Journal of Computational Mathematics, Vol.12, No.2, 1994, 163–172.

MONOTONE PIECEWISE CURVE FITTING ALGORITHMS*1)

Zhang Zheng-jun Yang Zi-qiang Zhang Chun-ming
(Computing Center, Academia Sinica, Beijing, China)

Abstract

A piecewise cubic curve fitting algorithm preserving monotonicity of the data without modification of the assigned slopes is proposed. The algorithm has the same order of convergence as Yan's algorithm[8] and Gasparo-Morandi's algorithm[5] for accurate or $O(h^q)$ accurate given data, but it has a more visually pleasing curve than those two algorithms. We also discuss the convergence order of cubic rational interpolation for $O(h^q)$ accurate data.

1. Introduction

The problem of monotonicity preserved interpolation has been considered by a number of authors. Fritsch and Carlson[4] have obtained necessary and sufficient conditions for a cubic Hermite interpolant to be monotone on an interval. Eisentat, Jackson and Lewis[3] derived a fourth-order accurate algorithm which is a modification of Fritsch and Carlson's algorithm. Beatson and Wolkowicz[1] considered monotone interpolation schemes of the fitting and modifying type, and gave the optimal order error properties of their algorithms. Gregory and Delbourgo[6] gave an explicit representation of a piecewise rational quadratic function; they also gave an explicit representation of a piecewise rational cubic function[2]; both explicit representations produce monotone interpolation for given monotone data. Yan[8] gave a piecewise cubic curve fitting algorithm without modification of the assigned slopes through inserting two knots to construct a horizontal line on a non-monotone interval. Gasparo-Morandi's algorithm[5] is a modification of Yan's algorithm[8], which inserts two knots to construct a slope line on a non-monotone interval.

Our algorithm which inserts two knots to construct two quadratic curves on a non-monotone interval is also a modification of Yan's algorithm[8] and Gasparo-Morandi's algorithm[5]. An $O(h^4)$ convergence result is obtained when the exact function and derivative values are available; otherwise, an $O(h^p)$ $(p = \min(4, q))$ convergence is obtained for an $O(h^q)$ accurate function and derivative values. The proof process of the main result is similiar to that in Yan[8] and Gasparo-Morandi[5]. We also discuss

* Received June 3, 1993.
1) The Project Supported by National Natural Science Foundation of China.

the convergence order of cubic rational interpolation with the $O(h^q)$ accurate function and derivative values, and an $O(h^p)$ convergence is obtained.

The paper begins with a definition of cubic interpolant, necessary and sufficient conditions, and construction of our algorithm. The convergence analysis of the algorithm is discussed in **3**. The convergence analysis of cubic rational interpolation for $O(h^q)$ accurate data is discussed in **4**. Finally, in **5**, examples applied with various interpolation methods and comparison are given.

2. The Algorithm

Let $f(x) \in C^1[a,b]$ be a monotone increasing function. Let $\pi: a = x_1 < x_2 < \cdots < x_n = b$ be a partition of the interval $I = [a,b]$. Suppose that y_i and d_i are approximate values of $f(x)$ and $f'(x)$ at the partition points x_i respectively. Let $h_i = x_{i+1} - x_i, \Delta y_i = y_{i+1} - y_i, \Delta_i = \Delta y_i / h_i, i = 1, 2, \cdots, n$. In particular we suppose that there exists an integer $q > 0$ such that

$$y_i = f(x_i) + O(h^q), \qquad d_i = f'(x_i) + O(h^q), \qquad i = 1, 2, \cdots, n \tag{2.1}$$

where $h = \max\{h_i\}$. Now, we construct a piecewise cubic function $s(x) \in C^1[I]$ such that

$$s(x_i) = y_i, \qquad s'(x_i) = d_i, \qquad i = 1, 2, \cdots, n. \tag{2.2}$$

In each subinterval $I_i = [x_i, x_{i+1}]$, s(x) is defined by

$$s_i(x) = \frac{d_i + d_{i+1} - 2\Delta_i}{h_i^2}(x - x_i)^3 + \frac{-2d_i - d_{i+1} + 3\Delta_i}{h_i}(x - x_i)^2 + d_i(x - x_i) + y_i. \tag{2.3}$$

It is clear that a necessary condition for monotonicity is that

$$\operatorname{sgn}(d_i) = \operatorname{sgn}(d_{i+1}) = \operatorname{sgn}(\Delta_i). \tag{2.4}$$

Furthermore, if $\Delta_i = 0$, then $s(x)$ is monotone (i.e. constant) on I_i if and only if $d_i = d_{i+1} = 0$. The remainder of this section assumes that $\Delta_i \neq 0$ and (2.4) is satisfied. Let $\alpha_i = d_i/\Delta_i, \beta_i = d_{i+1}/\Delta_i$. Then we have the following lemmas[4].

Lemma 1. *If $\alpha_i + \beta_i - 2 \le 0$, then $s(x)$ is monotone on I_i if and only if (2.4) is satisfied.*

Lemma 2. *If $\alpha_i + \beta_i - 2 > 0$, and (2.4) is satisfied, then $s(x)$ is monotone on I_i if and only if one of the following conditions is satisfied:*

(i) $2\alpha_i + \beta_i + 3 \le 0$,

(ii) $\alpha_i + 2\beta_i - 3 \le 0$, *or*

(iii) $\phi(\alpha_i, \beta_i) \ge 0$,

where $\phi(\alpha, \beta) = \alpha - (2\alpha + \beta - 3)^2 / 3(\alpha + \beta - 2)$.

In general, $s_i'(x)$ has the following form: $s_i'(x) = a(x - \bar{x})^2 + \omega$, where $\bar{x}$ is the extreme point of $s_i'(x)$. We denote μ, η, ω as $\mu = \bar{x} - x_i$, $\eta = x_{i+1} - \bar{x}$, $\omega = s_i'(\bar{x})$. It is clear that $s(x)$ is not monotone on I_i if and only if

$$0 < \mu, \quad \eta < h_i \quad \text{and} \quad \Delta_i \omega < 0. \tag{2.5}$$