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SINGULARITY AND QUADRATURE REGULARITY OF $(0, 1, \cdots, m-2, m)$-INTERPOLATION ON THE ZEROS OF JACOBI POLYNOMIALS*[1)]

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Abstract

In this paper we show that, if a problem of $(0, 1, \cdots, m-2, m)$-interpolation on the zeros of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta \geq -1)$ has infinite solutions, then the general form of the solutions is $f_0(x) + Cf(x)$ with an arbitrary constant C, where $f_0(x)$ and $f(x)$ are fixed polynomials of degree $\leq mn-1$. Moreover, the explicit form of $f(x)$ is given. A necessary and sufficient condition of quadrature regularity of the interpolation in a manageable form is also established.

1. Introduction and Main Results

Let us consider a system A of nodes

$$1 \geq x_1 > x_2 > \cdots > x_n \geq -1, \quad n \geq 2. \tag{1.1}$$

Let $\mathcal{P}_n$ be the set of polynomials of degree at most n and let $m \geq 2$ be a fixed integer. The problem of $(0, 1, \cdots, m-2, m)$-interpolation is, given a set of numbers

$$y_{kj}, \quad k \in N := \{1, 2, \cdots, n\}, \quad j \in M := \{0, 1, \cdots, m-2, m\}, \tag{1.2}$$

to determine a polynomial $R_{mn-1}(x; A) \in \mathcal{P}_{mn-1}$ (if any) such that

$$R_{mn-1}^{(j)}(x_k; A) = y_{kj}, \quad \forall k \in N, \quad \forall j \in M. \tag{1.3}$$

If for an arbitrary set of numbers y_{kj} there exists a unique polynomial $R_{mn-1}(x; A) \in \mathcal{P}_{mn-1}$ satisfying (1.3), then we say that the problem of $(0, 1, \cdots, m-2, m)$-interpolation on A is regular (otherwise, is singular) and $R_{mn-1}(x; A)$ can be uniquely written as

$$R_{mn-1}(x; A) = \sum_{\substack{k \in N \\ j \in M}} y_{kj} r_{kj}(x; A) \tag{1.4}$$

where $r_{kj} \in \mathcal{P}_{mn-1}$ satisfy

$$r_{kj}^{(\mu)}(x_\nu) = \delta_{k\nu}\delta_{j\mu}, \quad k, \nu \in N, \quad j, \mu \in M. \tag{1.5}$$

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In particular, for convenience of use we set

$$\rho_k(x) := r_{km}(x), \quad k = 1, 2, \cdots, n. \tag{1.6}$$

On the problem of $(0,2)$-interpolation Turán raises in [5] an open problem as follows.

Problem 29. *Find all Jacobi matrices $P(\alpha,\beta)$, $\alpha \neq \beta$, for which $(0,2)$-interpolation problem does have a unique solution.*

By a Jacobi matrix $P(\alpha,\beta)$, Turán means the triangular matrix whose nth row consists of the zeros of the nth Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ $(\alpha,\beta \geq -1)$.

Chak, Sharma, and Szabados [1] have given a necessary and sufficient condition of regularity of $(0,2)$-interpolation in a manageable form on all Jacobi matrices $P(\alpha,\beta)$.

Recently, the author generalized in [3] their important result and proved the following theorem, in which

$$\gamma := \frac{1}{2}(m-1)(\alpha+1), \quad \delta := \frac{1}{2}(m-1)(\beta+1), \tag{1.7}$$

$$s_k := 2^{-n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}, \quad k = 0, 1, \cdots, n. \tag{1.8}$$

Theorem A. *The problem of $(0,1,\cdots,m-2,m)$-interpolation on the zeros of $P_n^{(\alpha,\beta)}(x)$ $(\alpha,\beta \geq -1)$ is regular if and only if*

$$D_n(\alpha,\beta) \neq 0 \tag{1.9}$$

where

$$D_n(\alpha,\beta) = \begin{cases} \sum_{k=0}^{n} \dfrac{(-1)^k\binom{\gamma}{k}\binom{\delta}{n-k}s_k}{\binom{n}{k}}, & \alpha,\beta > -1, \\ (m+1)\dbinom{\delta}{n} - (m-1)\dbinom{n+\beta+\delta}{n}, & \alpha = -1, \ \beta > -1, \\ (-1)^n D_n(-1,\alpha), & \alpha > -1, \ \beta = -1, \\ 1 + (-1)^n, & \alpha = \beta = -1. \end{cases} \tag{1.10}$$

In particular, when $\alpha = -1$, $\beta > -1$ or $\alpha > -1$, $\beta = -1$, the problem is always regular; when $\alpha = \beta = -1$, the problem is regular for even n and singular for odd n.

Remark. In [3] we also gave the explicit forms for the fundamental polynomials, which are very complicated and omitted.

If the problem of $(0,1,\cdots,m-2,m)$-interpolation on A is not regular, then for a given set of numbers y_{kj} either there is no polynomial $R_{mn-1}(x)$ satisfying (1.3) or there is an infinity of polynomials with the property (1.3). The possibility of an infinity of solutions raises the question on the dimensionality of their number. The first result concerning this question, to the best of the author's knowledge, is given by Surányi and Turán in [4, Sections 6 and 11] for $\alpha = \beta = -1$: in the case of infinitely many solutions for $n \geq 3$ the general form of the solutions is

$$f(x) = f_0(x) + C\pi_n(x)[P_{n-1}(x) - 3]$$