

THE RANK- k UPDATING ALGORITHM FOR THE EXACT INVERSION OF MATRICES WITH INTEGER ELEMENTS*

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Abstract

In this paper, the numerical solution of the matrix problems over a ring of integers is discussed. The rank- k updating algorithm for the exact inversion of a matrix is proposed. This algorithm is generally more effective than Jordan elimination. The common divisor of the numbers involved is reduced to avoid over-swelling of intermediate numbers.

§1. Introduction

Solving such problems as arising in number theory, graph theory, coding theory, statistics, linear programming and circuit theory require computing the exact inversion of a matrix with integer elements. The exact solution of a problem of a matrix also plays an important role in numerical experiments. For example, when matrices are ill-conditioned, the integral methods are more interesting. What is most difficult in integer computation is the rapid increase of the numbers involved when the usual algorithms are applied. So, first of all, an integer algorithm must be able to restrain intermediate numbers from swelling. Many algorithms have been developed^[2-6], such as

- (a) Elementary transformations method;
- (b) Division-free Gaussian elimination;
- (c) One-step Jordan elimination;
- (d) Multistep Gaussian elimination;
- (e) Congruence method;
- (f) Determinant and Gramer's rule.

In this paper, a rank- k updating algorithm for the exact inverse matrix and the exact solution of a system of linear equations is proposed. This algorithm is generally more effective than the Jordan elimination. The common divisors of the numbers involved are reduced to restrain intermediate numbers from swelling.

§2. The Rank- k updating algorithm

Suppose that A is a nonsingular n by n integer matrix and u, v are n by k rank- k integer matrices. Then uv^t is a rank- k matrix and $A + uv^t$ is a rank- k updating matrix.

* Received December 25, 1987.

It is easy to prove that the matrix $A + uv^t$ satisfies the following identical relation:

$$\det (A + UV^t) = \det (A) \det (I + v^t A^{-1} u). \quad (2.1)$$

The inversion of matrix $A + uv^t$ can be represented by the following formula if the determinant of $A + uv^t$ is non-zero:

$$(A + uv^t)^{-1} = A^{-1} - A^{-1} u (I + v^t A^{-1} u)^{-1} v^t A^{-1}. \quad (2.2)$$

Define the following notation:

$$n = m * k, \quad A = (a_1, a_2, \dots, a_n)^t, \quad A_0 = I,$$

$$U_i = (e_{ik+1}, e_{ik+2}, \dots, e_{(i+1)k}), \quad \tilde{V}_i = (a_{ik+1}, a_{ik+2}, \dots, a_{(i+1)k}), \quad V_i = \tilde{V}_i - U_i,$$

where I is an identity matrix, with i -th column as e_i . With this notation, the following representations are obtained:

$$A_i = A_{i-1} + u_{i-1} v_{i-1}^t, \quad A = A_m, \quad i = 1, 2, \dots, m. \quad (2.3)$$

The exact $\det (A)$ and $\det (A)A^{-1}$ are computed by determining the sequence of $\det (A_i)$ and $\det (A_i)A_i^{-1}$. Formula (2.2) shows that

$$A_1^{-1} = (I + u_0 v_0^t)^{-1} = I - u_0 (I + v_0^t u_0)^{-1} v_0^t. \quad (2.4)$$

Let

$$d_1 = \det(I + v_0^t u_0), \quad w_1 = d_1 (I + v_0^t u_0)^{-1}. \quad (2.5)$$

From equation(2.1), $d_1 = \det (A_1)$. If d_1 is non-zero, the relation (2.4) holds. It follows that

$$d_1 A_1^{-1} = d_1 I - u_0 w_1 v_0^t. \quad (2.6)$$

The matrix $d_1 A_1^{-1}$ is an adjoint matrix of A_1 , so the elements of the matrix are integers. Generally, there is no common divisor in the matrix, and the same is true with matrix w_1 . From relation (2.2),

$$\begin{aligned} A_2^{-1} &= (A_1 + u_1 v_1^t)^{-1} = A_1^{-1} - A_1^{-1} u_1 (I + v_1^t A_1^{-1} u_1)^{-1} v_1^t A_1^{-1} \\ &= A_1^{-1} - A_1^{-1} u_1 d_1 (d_1 I + v_1^t d_1 A_1^{-1} u_1)^{-1} v_1 A_1^{-1} \end{aligned} \quad (2.7)$$

Let $d_2 = \det (A_2)$. Apply relation (2.1) again to obtain

$$d_2 = \det (d_1 I + v_1^t d_1 A_1^{-1} u_1). \quad (2.8)$$

If w_2 is defined as

$$w_2 = d_2 (d_1 I + v_1^t d_1 A_1^{-1} u_1)^{-1}, \quad (2.9)$$