Journal of Computational Mathematics, Vol.9, No.1, 1991, 5–8.

ESTIMATE OF CONDITION NUMBER FOR SOME DISCRETE ILL-POSED EQUATIONS*

Tang Long-ji
(*Hunan Computing Center, Changsha, China*)

Abstract

This paper extends the numerical method and the estimate of the condition number for some discrete ill-posed equations with positive definite matrix in [2] to the case with generalized positive definite matrix.

§1. Introduction

This paper considers the following discrete ill-posed problem:

$$Ax = b \tag{1.1}$$

where A belongs to the set

$$D = \{A \in R^{n\times n} | (Ax, x) > 0, x \in R^n \setminus 0\}, \tag{1.2}$$

and the data b and solution x lie in the n-dimensional vector space R^n. In practice, this problem is ill-posed as A has a large condition number, which is defined as the radio of the largest to the smallest singular values. Here a bounded inverse of A does exist in theory, but the solution $x = A^{-1}b$ is numerically unstable.

Let b_δ be approximate or measured data satisfying

$$\|b_\delta - b\|_2 \le \delta, \tag{1.3}$$

where $\|\cdot\|_2$ is Euclid's norm, $\delta \ge 0$. Here we can replace equation (1.1) by an approximate equation with prior estimate (1.3)

$$Ax = b_\delta. \tag{1.4}$$

For the ill-posed problem (1.4), Tikhonov [1] et al. have developed several very useful numerical methods based on the least-squares principle:

$$\|Ax - b_\delta\|_2^2 + \alpha^2\|x\|_2^2 = \min. \tag{1.5}$$

*Received December 5, 1989.

The minimal solution is

$$x_\alpha = (A^*A + \alpha^2 I)^{-1} A^* b_\delta. \tag{1.6}$$

In [2], J.N. Franklin has analyzed a different numerical method. (1.4) is replaced by the following approximate equation

$$(A + \alpha I)x_\alpha = b_\delta, \tag{1.7}$$

in which $\alpha > 0$, A is a positive definite matrix and I is the unit matrix. The solution by Franklin's method has a simple form

$$x_\alpha = (A + \alpha I)^{-1} b_\delta. \tag{1.8}$$

Above-mentioned Franklin's method is generally less applicable than Tikhonov's, because it applies only to the ill-posed problem $Ax = b$ in which A is a positive definite operator. In this paper, Franklin's method will be extended to the discrete ill-posed problem (1.4) in which A belongs to set D. And we will give an estimate of the condition number for (1.6) and (1.8) for $A \in D$.

We now define $\|\cdot\|_2$ to be the spectral norm, and $K(A)$ the spectral condition number of matrix A. Moreover, we define several sets:

$$D_1 = \{B \in R^{n\times n} | B \text{ is a positive definite matrix}\},$$
$$D_2 = \{B \in R^{n\times n} | B^* = B\}.$$

§2. Estimate of Condition Number

Although Franklin's method is less applicable than Tikhonov's, its simple form of solution has many advantages in numerical analysis. [2] gives the following estimate of the condition number for the two methods:

$$\frac{1}{2} \le \frac{K^2(A + \alpha I)}{K(A^*A + \alpha^2 I)} \le 2, \tag{2.1}$$

where $A \in D_1$.

(2.1) implies that Franklin's method is not only simpler but also stabler than Tikhonov's as A belongs to D_1. In the following, we will give an estimate of the condition number similar to (2.1) when A belongs to D.

Let $\{\lambda_i(A)\}, \{\sigma_i(A)\}, i = 1, 2, \cdots, n$, be the eigenvalues and the singular values of matrix A, respectively. And we assume

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n. \tag{2.2}$$

If $A \in D_2$, then suppose

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n. \tag{2.3}$$

Lemma 1. *If $A \in D$, then $A + A^* \in D_1$.*