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APPROXIMATE SOLUTIONS AND ERROR BOUNDS FOR SOLVING MATRIX DIFFERENTIAL EQUATIONS WITHOUT INCREASING THE DIMENSION OF THE PROBLEM*

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Abstract

In this paper we present a method for solving initial value problems related to second order matrix differential equations. This method is based on the existence of a solution of a certain algebraic matrix equation related to the problem, and it avoids the increase of the dimension of the problem for its resolution. Approximate solutions, and their error bounds in terms of error bounds for the approximate solutions of the algebraic problem, are given.

§1. Introduction

Second order matrix differential equations of the type

$$X^{(2)}(t) + A_1 X^{(1)}(t) + A_0 X(t) = F(t); \quad X(0) = C_0, \quad X^{(1)}(0) = C_1 \tag{1.1}$$

where A_i, C_i, for $i = 0, 1$, and $F(t)$ are square complex matrices, elements of $\mathbb{C}_{p\times p}$, and F is continuous, appear in the theory of damped oscillatory systems and vibrational systems[6].

The standard method for solving equations of the type (1.1) is based on the consideration of the change $Y_1 = X; Y_2 = X^{(1)}$, and the equivalent first order extended linear system

$$\left(\frac{d}{dt}\right)\begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = C_L \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ F(t) \end{bmatrix}; \quad C_L = \begin{bmatrix} 0 & I \\ -A_0 & -A_1 \end{bmatrix} \tag{1.2}$$

and the solution of problem (1.1) is given by

$$X(t) = [I, 0]\left\{ \exp(tC_L) \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} + \int_0^t \exp((t-s)C_L) \begin{bmatrix} 0 \\ F(s) \end{bmatrix} \right\} ds. \tag{1.3}$$

The expression (1.3) for the solution of problem (1.1) has the inconvenience of the increase of the dimension of the problem and the aim of this paper is to present a method for solving (1.1), without increasing the dimension of the original problem, which provides approximate solutions and their error bounds in terms of data and a solution of the algebraic equation

$$X^2 + A_1 X + A_0 = 0. \tag{1.4}$$

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Explicit methods for solving equations of the type (1.4) may be found in [1, 4], and iterative methods for its resolution are given in [9–12, 14].

For the sake of clarity in the presentation of the paper we recall some concept and properties that will be used below. If A, B are matrices in $\mathbb{C}_{p\times p}$, we denote by $\| \; \|$ the operator norm that is defined by the expression

$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$

where for a vector y in $\mathbb{C}_p$, the symbol $\|y\|$ means the Euclidean norm of y. From [3] and [8], it follows that

$$\|AB\| \leq \|A\| \; \|B\|$$

and

$$\| \exp(tA) - \exp(tB)\| \leq \exp(t\|B\|)(\exp(t\|A-B\|) - 1) \tag{1.5}$$

where t is a real number.

§2. Approximate Solutions and Error Bounds

We begin this section with a result that provides a sequence of approximations that converges to the unique solution of problem (1.1), without increasing the dimension of the problem and under the existence hypothesis of a solution of the algebraic equation (1.4).

Theorem 1. *Let X_0 be a solution of equation (1.4), and let $\{Z_n\}_{n\geq 1}$ be a sequence of matrices in $\mathbb{C}_{p\times p}$ that converges to X_0 in the operator norm, and let us suppose that F is a continuous function. The sequence of matrix functions $X_n(t)$, defined by*

$$X_n(t) = \exp(tZ_n)C_n(t) + \Big(\int_0^t \exp((t-s)Z_n)\exp(-s(Z_n + A_1))ds\Big) D_n(t), \tag{2.1}$$

$$C_n(t) = C_0 - \int_0^t \int_0^s \exp(-uZ_n)\exp((-u+s)(Z_n + A_1))F(s)duds,$$

$$D_n(t) = (C_1 - Z_nC_0) + \int_0^t \exp(s(Z_n + A_1))F(s)ds \tag{2.2}$$

where $n \geq 1$, is pointwise convergent to the unique solution of problem (1.1), given by

$$X(t) = \exp(tX_0)C(t) + \Big(\int_0^t \exp((t-s)X_0)\exp(-s(X_0 + A_1))ds\Big) D(t), \tag{2.3}$$

$$C(t) = C_0 - \int_0^t \int_0^s \exp(-uX_0)\exp((-u+s)(X_0 + A_1))F(s)duds,$$

$$D(t) = C_1 - X_0C_0 + \int_0^t \exp(s(X_0 + A_1))F(s)ds. \tag{2.4}$$

Proof. It is clear that $X_1(t) = \exp(tX_0)$ is a solution of the homogeneous operator differential equation

$$X^{(2)} + A_1X^{(1)} + A_0X = 0. \tag{2.5}$$