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POLYNOMIAL ACCELERATION METHODS FOR SOLVING SINGULAR SYSTEMS OF LINEAR EQUATIONS*1)

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Abstract

In this paper we study the polynomial acceleration methods for solving singular linear systems. We establish iterative schemes, show their convergence and find iteration error bounds.

§1. Introduction

For many practical problems, such as Neumann problems and those for elastic bodies with three surfaces and Poisson's equation on a sphere and with periodic boundary conditions, their finite difference and finite element formulations lead to singular but consistent systems of linear equations. In addition, when an eigenvalue problem is solved by a relaxation method, the solution of a singular linear system is involved[9]. However, as pointed in [1], methods for solving singular systems of linear equations have unfortunately been somewhat neglected in literature. Perhaps this is due to some of the difficulties involved in establishing criteria for convergence.

In this paper we study polynomial acceleration methods for solving singular linear systems. We establish iterative schemes, show their convergence and find iteration error bounds.

For convenience, we discuss real systems. All results obtained in this paper can be easily generalized to complex systems.

We use the following notations: E^n is an n-dimensional real vector space, $E^{n\times n}$ stands for a set of all $n\times n$ real matrices, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ represent null space and column space (range of value) of matrix A, respectively, $\sigma(A)$ stands for the set of all eigenvalues of matrix A and A^T and A^+ are the transpose and the Moore-Penrose inverse of matrix A, respectively.

§2. Basic Iterative Methods

Consider a linear system

$$Ax = b, \tag{2.1}$$

where $A\in E^{n\times n}, x\in E^n$ and $b\in \mathcal{R}(A)$. We construct a (linear stationary) basic iterative method

$$x^{\nu+1} = Tx^{\nu} + g, \tag{2.2}$$

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where the iterative matrix $T \in E^{n\times n}, g \in E^n, x^\nu, x^{\nu+1} \in E^n$. (2.2) can be written as follows:

$$x^{\nu+1} = x^\nu - H(Ax^\nu - b), \tag{2.3}$$

where $H \in E^{n\times n}$. From (2.2) and (2.3) we have

$$T = I - HA, \quad g = Hb. \tag{2.4}$$

Let $x^*(x^0)$ denote a solution of (2.1) which is a limit of a vector sequence produced by an iterative method (not necessarily a linear stationary iterative method) with x^0 as an initial iterative vector. Then we define the set of error vectors U[8]:

$$U = \{y : y = x - x^*(x), \quad x \in U_0\}, \tag{2.5}$$

where U_0 is a set of the initial iterative vectors. When U is a subspace of E^n, we use $\|\cdot\|_U$ to denote a vector norm in U and the induced matrix norm. Then we use $R_\infty(\tilde{T})$,

$$R_\infty(\tilde{T}) = -\lim_{\nu\to\infty} \frac{1}{\nu} \ln \|\tilde{T}^{(\nu)}\|_U, \tag{2.6}$$

to denote the asymptotic rate of convergence of an iterative method for solving singular systems, where $\tilde{T}^{(\nu)}$ stands for the error transition operator of the ν-th iteration[8].

If we consider the linear stationary iterative method (2.2)–(2.4) and introduce the sub-spectral radius of the iterative matrix T:

$$\gamma(T) = \max\{|\lambda| : \lambda \in \sigma(T) \cup \{0\} \backslash \{1\}\}, \tag{2.7}$$

then we have the following result:

Theorem 2.1[8]. *The linear stationary iterative method* (2.2)–(2.4) *is convergent in* $U_0 \equiv E^n$ *if and only if*

(i) $\gamma(T) < 1$,

(ii) rank $(I - T)$ = rank $(I - T)^2$,

(iii) $\mathcal{N}(A) = \mathcal{N}(HA)$ *or, equivalently,* $\mathcal{N}(H) \cap \mathcal{R}(A) = \{0\}$.

When the linear stationary iterative method is convergent, U *(cf.* (2.5)) *must be a subspace:* $U = \mathcal{R}(HA)$, *and the asymptotic rate of convergence is*

$$R_\infty(T) = -\ln \gamma(T). \tag{2.8}$$

Note that the basic iterative method (2.2)–(2.4) can be derived from the splitting of matrix A:

$$A = H^+ - H^+T. \tag{2.9}$$

Definition 2.1. *The iterative method* (2.2) *is symmetrizable if for some nonsingular matrix* W *the matrix* $W(I - T)W^{-1}$ *is symmetric positive semidefinite (SPSD). Such a matrix* W *is called a symmetrization matrix.*

Obviously, if the iterative method (2.2) is symmetrizable, then the eigenvalues of T are real and matrix T is diagonalizable. Hence the condition (ii) of Theorem 2.1 is satisfied. Let

$$m(T) = \min\{\lambda : \lambda \in \sigma(T)\}, \quad M(T) = \max\{\lambda : \lambda \in \sigma(T)\};$$

then we have

$$M(T) \le 1. \tag{2.10}$$