ON BOUNDARY INTEGRAL EQUATIONS OF THE FIRST KIND [1)]

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Abstract

A large class of elliptic boundary value problems in elasticity and fluid mechanics can be reduced to systems of boundary integral equations of the first kind. This paper summarises some of the basic concepts and results concerning the mathematical foundation of boundary element methods for treating such a class of boundary integral equations.

§1. Introduction

The boundary integral equation method for numerical solutions to elliptic boundary value problems has received much attention and gained wide acceptance in recent years. As is well known, the method is particularly suitable for obtaining numerical solutions of exterior boundary value problems and implies an approximate technique by which the problem dimensions are reduced by one. The latter leads to an appreciable reduction in the numbers of algebraic equations generated for solutions, as well as much simplified data presentation. However, irrespective of the particular numerical implementation chosen, central to the method is the reduction of boundary value problem to equivalent boundary integral equations over the boundary of the domain for the problems under consideration. This reduction is by no means unique.

In the conventional approach, Fredholm integral equations of the second kind are generally obtained either by using the "direct method" based on Green's formula or the "indirect method" in which case solutions are expressed in terms of simple or double layer potentials depending on the problem under consideration. The integral equations of the second kind are numerically stable and hence have been used extensively in engineering applications. However, in contrast to the partial differential

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equations, all the essential properties of the original elliptic operators such as symmetry and coerciveness are generally not preserved for the corresponding boundary integral operators in the variational formulation. Hence from the theoretical and computational points of view, the boundary element method for the Fredholm integral equations of the second kind is not satisfactory.

Alternatively, for a variety of physical problems with the Dirichlet data, if one expresses the solutions in terms of simple-layer potentials (Fichera[3], Fichera and Ricci[4], Hsiao[10–11], Hsiao and MacCamy [21–22], Hsiao and Wendland[25–30], LeRoux[31], and MacCamy[32]) or employs Green's formula for the solutions (Hsiao and Roach[24], Nedelec and Planchard[33]), boundary integral equations of the first kind will result. Similarly for problems with the Neumann data, boundary integral equations of the first kind (involving hypersingular integral operators) can be obtained by using double–layer potentials or by differentiating the Green representation formula for the solutions (Feng[6–7], Giroire and Nedelec[8], Han[9], Hsiao[16], Hsiao and Wendland[27, 29–30], and Wendland[36]). In these formulations, in contrast to the integral equations of the second kind, the symmetry and coerciveness properties of the integral operators follow directly from those of the original partial differential operators via the trace theorem in Sobolev spaces and vice versa. Hence the boundary element method for the integral equations of the first kind is more satisfactory and compatible with the finite element method for the partial differential equations.

In this paper, for simplicity we will confine to the model problems for the Laplacian in $I\!R^n, n = 2, 3$, the exterior Dirichlet and Neumann Problems. In either case we will reduce the boundary-value problem to a boundary integral equation of the first kind via the direct or indirect method. As will be seen, the corresponding boundary integral operators are typical pseudodifferential operators of order 2α where α is equal to $-\frac{1}{2}$ for the Dirichlet problem and $+\frac{1}{2}$ for the Neumann problem.

§2. Boundary Integral Equations

Throughout the paper, let Γ be a sufficiently smooth simple closed curve in $I\!R^2$ or surface in $I\!R^3$, and let Ω^c be the exterior domain. We consider two fundamental boundary value problems for the Laplace equation

$$\Delta u = 0 \quad \text{in} \quad \Omega^c, \tag{1}$$

the *Dirichlet problem* (D) and the *Neumann problem* (N). The boundary conditions and the conditions at infinity are

$$u|_\Gamma = \phi \quad \text{on} \quad \Gamma \quad \text{and} \quad u = O(|x|^{2-n}) \quad \text{as} \quad |x| \to \infty \tag{2}$$