AN INTEGRAL METHOD FOR CONSTRUCTING BIVARIATE SPLINE FUNCTIONS * 1)

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Because of the variety of partitions which arise in higher dimensions, there so far is no unified theory about multivariate spline functions. Briefly, we have two strategies at present. One is the so-called classical approach. Since the work by one author of this paper in 1975 [1], this method has been producing many results. Some dimension formulas, expressions of B-splines and bases of some spline spaces on certain partitions were obtained (cf [2], [3]). But calculations are complicated in the case of B-splines, as both smoothness and locally supported conditions must be taken into account.

The second strategy is the polyhedral spline approach. This idea which originated with a geometric interpretation of the univariate B-splines and multivariate splines was obtained as the volume of slices of a polyhedron. Many results such as linear independence, approximation rates and properties of spline spaces have been gained. Choosing various kinds of polyhedra, we have various kinds of splines. From recurrence relations in [6], it is possible to get expressions of those splines, though the quantity of calculations is rather large and polyhedral splines sometimes have large supports.

In this paper, we will give an integral method to construct spline functions, trying to link up two strategies mentioned above. We will show the integral recursions for splines on uniform partitions. If the original spline has a minimal support, it is possible to produce minimal supported splines with more smoothness by the integral method, that is, we provide recursions for B-splines. As spline spaces with maximal smoothness which include B-splines are more useful, we also give bases of these spaces consisting of B-splines and truncated power functions. For the sake of clarity, we only discuss the case of two dimensions. The results can be applied to higher dimensions without any virtual difficulty.

We introduce some notations first. The partitions Δ_1:

$$x = i, y = j, x - y = k \text{ and } \Delta_2 : x = i, y = j, x - y = k, x + y = k,$$

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$i, j, k = 0, \pm 1, \pm 2, \cdots$, are called cross-cut triangulation and criss-cross triangulation of uniform rectangular partitions, respectively. Let

$$r'_{ij} = \left\{(x,y) : i \le x \le i+1,\ j \le y \le j+1,\ x-i \le y-j\right\},$$
$$r''_{ij} = \left\{(x,y) : i \le x \le i+1,\ j \le y \le j+1,\ x-i \ge y-j\right\},$$

which are cells of Δ_1. Let

$$\omega_{ij}^{(1)} = \left\{(x,y) : i \le x \le i+1, j \le y \le j+1, x-i \le y-j, x-i \le j+1-y\right\},$$
$$\omega_{ij}^{(2)} = \left\{(x,y) : i \le x \le i+1, j \le y \le j+1, x-i \le y-j, x-i \ge j+1-y\right\},$$
$$\omega_{ij}^{(3)} = \left\{(x,y) : i \le x \le i+1, j \le y \le j+1, x-i \ge y-j, x-i \ge j+1-y\right\},$$
$$\omega_{ij}^{(4)} = \left\{(x,y) : i \le x \le i+1, j \le y \le j+1, x-i \ge y-j, x-i \le j+1-y\right\},$$

which are cells of Δ_2. If k, μ are nonnegative integers, $S_k^\mu(\Delta_i), i = 1, 2$, is a space of bivariate pp functions in C^μ of degree k, that is

$$S_k^\mu(\Delta_i) = \left\{s(x,y) : s(x,y) \in C^\mu(R^2); s(x,y) \in P_k, \text{ when } (x,y) \in \text{ cells of } \Delta_i\right\}, \quad i = 1, 2.$$

If $B(x,y) \in S_k^\mu(\Delta_i), i = 1, 2$, and T is a bounded region in R^2 so that $B(x,y) > 0$ when $(x,y) \in \text{int}\ (T)$ and $B(x,y) = 0$ when $(x,y) \bar{\in} \text{ int } (T)$, we call $B(x,y)$ a locally supported spline for short. By B-spline we denote a locally supported spline with a minimal support.

From [2] we know that a necessary condition for the existence of a nontrivial locally supported spline in $S_k^\mu(\Delta_1)$ is that k, μ satisfy the inequality $k \ge (3\mu+1)/2$, and in $S_k^\mu(\Delta_2), k > (4\mu+1)/3$. We will prove that they are also sufficient conditions. Let d be the smallest integer satisfying $d > (3\mu+1)/2$ (for $\Delta_2, d > (4\mu+1)/3$); then $S_d^\mu(\Delta_i), i = 1, 2$, are spline spaces in $C^\mu(D)$ with the lowest degree which contain B-splines. The existence and recursions of B-splines in $S_d^\mu(\Delta_i)$ and in ordinary spaces will be investigated. Box-splines in $S_d^\mu(\Delta_i)$ may not have minimal supports (cf [4]), so our integral method is somehow superier to the box-spline method. Besides, we will show two examples to treat splines on non-uniform partitions with the integral method. Finally, for rectangle D and refinement $\Delta_{mn}^{(i)}$ of partition Δ_i on D, we will make bases of $S_d^\mu(\Delta_{mn}^{(i)}, D)$ consisting of mainly B-splines.

§1. Existence and Construction of B–Splines in $S_d^\mu(\Delta_1)$

Lemma 1.1 *If $B(x;y)$ is a locally supported spline in $S_d^\mu(\Delta_1)$ with support T* (shown in Fig.1) *and $A(T) = (a_1, a_2, a_3, a_4, a_5, a_6)$ where $a_i, i = 1, 2, \cdots, 6$, denotes the number of*