THE WAVE EQUATION APPROACH TO ROBBIN INVERSE PROBLEMS FOR A DOUBLY-CONNECTED REGION: AN EXTENSION TO HIGHER DIMENSIONS*

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Abstract

The spectral function $\hat{\mu}(t) = \sum_{j=1}^{\infty} e^{-it\lambda_j^{1/2}}$ where $\{\lambda_j\}_{j=1}^{\infty}$ are the eigenvalues of the three-dimensional Laplacian is studied for a variety of domains, where $-\infty < t < \infty$ and $i = \sqrt{-1}$. The dependence of $\hat{\mu}(t)$ on the connectivity of a domain and the impedance boundary conditions (Robbin conditions) are analysed. Particular attention is given to the spherical shell together with Robbin boundary conditions on its surface.

1. Historical Remarks

Let $D \subseteq R^3$ be a simply connected bounded domain with a smooth bounding surface S. Then, there exist eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ and corresponding eigenfunctions $\{\phi_j(\underset{\sim}{x})\}_{j=1}^{\infty}$ of the Laplace operator $\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in xyz-space, under the impedance boundary conditions (Robbin boundary conditions), such that $\{\phi_j(\underset{\sim}{x})\}_{j=1}^{\infty}$ is a complete orthonormal system in $L^2(D)$. That is, we have the following impedance problem (Robbin problem):

$$-\Delta_3\phi_j = \lambda_j\phi_j \quad \text{in } D, \tag{1.1}$$

$$(\frac{\partial}{\partial n} + \gamma)\phi_j = 0 \quad \text{on } S, \tag{1.2}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the inward pointing normal to S and γ is a positive constant. We may assume that each ϕ_j is real-valued and that the eigenvalues λ_j are enumerated in the order of magnitude

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_j \le \cdots \to \infty \quad \text{as } j \to \infty. \tag{1.3}$$

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There are numerous works treating the asymptotic behaviour of the number of eigenvalues, $N(\lambda)$, as $\lambda \to \infty$. It has been shown that (H. Weyl, 1912)

$$N(\lambda) \sim \frac{V}{6\pi^2}\lambda^{3/2} \quad \text{as } \lambda \to \infty, \tag{1.4}$$

and that (R. Courant, 1920)

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^{3/2} + O(\lambda \log \lambda) \quad \text{as } \lambda \to \infty, \tag{1.5}$$

where V is the volume of D.

In order to obtain further information about the geometry of D, one studies certain functions of the spectrum. The most useful to date comes from the study of the heat equation or the wave equation.

Accordingly, let $e^{-t\Delta_3}$ denote the heat operator. Then, we can construct the trace function

$$\theta(t) = \operatorname{tr}(e^{-t\Delta_3}) = \sum_{j=1}^{\infty} e^{-t\lambda_j}, \tag{1.6}$$

which converges for all positive t.

Suppose that $e^{-it\Delta_3^{1/2}}$ is the wave operator. Then an alternative to (1.6) is to study the trace function

$$\hat{\mu}(t) = \operatorname{tr}(e^{-it\Delta_3^{1/2}}) = \sum_{j=1}^{\infty} e^{-it\lambda_j^{1/2}}, \tag{1.7}$$

which represents a tempered distribution for $-\infty < t < \infty$ and $i = \sqrt{-1}$. The applications of (1.6) to problem (1.1) and (1.2) and to more general ones can be found in Gottlieb [1], Pleijel [4], Waechter [5], Zayed [6, 7] and the references given there. thus, Pleijel has investigated problem (1.1)–(1.2) by using the heat equation approach and has shown that: if $\gamma \to \infty$ (Dirichlet problem),

$$\theta(t) = \frac{V}{(4\pi t)^{3/2}} - \frac{S}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}} \int_S H ds + O(t^{1/2}) \quad \text{as } t \to 0, \tag{1.8}$$

and if $\gamma = 0$ (Neumann problem),

$$\theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{S}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}} \int_S H ds + O(t^{1/2}) \quad \text{as } t \to 0, \tag{1.9}$$

where V and S are respectively the volume and the surface area of the domain D while $H = \frac{1}{2}(\frac{1}{R_1} + \frac{1}{R_2})$, R_1 and R_2 are the principal radii of curvature.

Zayed [7] has investigated problem (1.1)– (1.2) for either large or small impedance γ, by using the heat equation approach, and has shown that, if $\gamma >> 1$,

$$\theta(t) = \frac{V}{(4\pi t)^{3/2}} - \frac{1}{16\pi t}\{S - 2\gamma^{-1}\int_S H ds\} + \frac{1}{12\pi^{3/2}t^{1/2}} \int_S H ds + O(t^{1/2}) \quad \text{as } t \to 0, \tag{1.10}$$