AN ACCELERATION METHOD IN THE HOMOTOPY NEWTON'S CONTINUATION FOR NONLINEAR SINGULAR PROBLEMS*

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Abstract

The nonlinear singular problem $f(u)=0$ is considered. Here f is a C^3 mapping from E^n to E^n. The Jacobian matrix $f'(u)$ is singular at the solution u^* of $f(u)=0$. A new acceleration method in the homotopy Newton's continuation is proposed. The quadratic convergence of the new algorithm is proved. A numerical example is given.

§ 1. Introduction

We consider the nonlinear singular problem

$$f(u)=0. \tag{1.1}$$

Here f is a C^3 mapping from E^n to E^n and u^* is a singular solution of (1.1), i.e. $f(u^*)=0$ and the Jacobian matrix $f'(u^*)$ is singular.

Newton's method and its acceleration in the neighborhood of a singular solution have been studied by many authors (see [2]—[9], [11], [13]—[15] for details), under the requirement that the initial guess not only is near u^* but also belongs to a special cone

$$W(\rho, \theta)=\{u\,|\,0<\|u-u^*\|<\rho,\ \|P_x(u-u^*)\|\leqslant\theta\|P_N(u-u^*)\|\}$$

for small ρ, θ, where N is the null space of $f'(u^*)$, X is the complement of N in E, P_N is the projection onto N and P_x is the projection onto X.

We assume the dimension of N is one, i.e. rank $f'(u^*)=n-1$. This is the case we usually meet. Denote

$$N=\{\alpha\phi\,|\,\alpha\in R\},\quad \phi\in E^n,\ \phi\neq 0,$$

$$M=\mathrm{Range}(f'(u^*))=\{y\in E^n\,|\,\psi y=0\},\quad \psi\in E^n,\ \psi\neq 0.$$

We introduce a homotopy continuation mapping $G(u, \lambda)=f(u)-\lambda f(u^0)$ from E^{n+1} to E^n. A point $(u, \lambda)\in E^{n+1}$ is called a regular point for G if DG: $E^{n+1}\to E^n$ is surjective. A point $v\in E^n$ is a regular value of G if each point of $G^{-1}(v)$ is a regular point for G.

Our idea is to transform the singularity in the original problem into the singularity in a scalar equation which is simply treated by an acceleration method. Compared with the other algorithms ours does not require that the initial guess must lie in a special cone $W(\rho, \theta)$ for small ρ, θ. Also, some combination of our

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algorithm with the other algorithms is possible. The initial guess for other algorithms can be obtained by our method.

§ 2. Pseudo-Arclength Continuation

We construct a homotopy

$$G(u, \lambda)=f(u)-\lambda f(u^0), \tag{2.1}$$

where u^0 is chosen in such a way that 0 is a regular value of $G(u, \lambda)$. According to Lemma 2.15 in [10], we can choose such u^0 with probability one.

Our purpose is to find a path $u(\lambda)$ from $\lambda=1$ to $\lambda=0$. Obviously $u(1)=u^0$, and $u(0)$ is just a solution of $f(u)=0$ that we want to solve. An auxiliary equation introduced in the pseudo-arclength continuation method is

$$N(u, \lambda; \sigma)=\dot{u}_*^T(u-u_*)+\dot{\lambda}_*(\lambda-\lambda_*)-(\sigma-\sigma_*), \tag{2.2}$$

where (u_*, λ_*) is a point on the homotopy path at $\sigma=\sigma_*$, $\dot{u}_*=du(\sigma_*)/d\sigma$, $\dot{\lambda}_*=d\lambda(\sigma_*)/d\sigma$, $\dot{u}_*^T$ is the transpose of $\dot{u}_*$.

§ 3. Computing the Root σ^* of $\lambda(\sigma)=0$

In order to get the solution of $f(u)=0$ we are concerned only with the root σ^* of $\lambda(\sigma)=0$ and the corresponding computation for $u(\sigma^*)$, rather than the whole homotopy path $\Gamma(\sigma)$: $[\lambda(\sigma), u(\sigma)]$.

Keller [9] proposed the secant iteration

$$\sigma_{j+1}=\sigma_j-\frac{\sigma_j-\sigma_{j-1}}{\lambda(\sigma_j)-\lambda(\sigma_{j-1})}\cdot\lambda(\sigma_j) \tag{3.1}$$

after σ_0 and σ_1, which satisfy $\lambda(\sigma_0)\cdot\lambda(\sigma_1)<0$, are computed. Of course we can use Newton's iteration for $\lambda(\sigma)=0$,

$$\sigma_{j+1}=\sigma_j-\lambda(\sigma_j)/\dot{\lambda}(\sigma_j). \tag{3.2}$$

The practical computations show that both methods converge slowly in our singular case because of

Theorem 1. *Along with the homotopy path* $\Gamma(\sigma)$: $[u(\sigma), \lambda(\sigma)]$, $\dot{\lambda}(\sigma^*)=0$ *if* $\lambda(\sigma^*)=0$.

Proof. u^0 was chosen in Section 1 such that

$$DG(u, \lambda)=(f'(u), f(u^0)) \tag{3.3}$$

is a surjective mapping from E^{n+1} to E^n. So

$$\operatorname{Rank}(f'(u), f(u^0))=n \quad \forall(u, \lambda)\in\Gamma. \tag{3.4}$$

Noticing

we have $$\operatorname{Rank} f'(u^*)=n-1 \text{ at } \sigma=\sigma^*$$

$$f(u^0)\,\overline{\in}\,\operatorname{Range} f'(u^*).$$

Otherwise $f(u^0)$ is a linear combination of each column of the matrix $f'(u^*)$, and therefore

$$\operatorname{Rank}(f'(u^*), f(u^0))=\operatorname{Rank} f'(u^*)=n-1.$$

That contradicts (3.4).