INTERVAL ITERATIVE METHODS UNDER PARTIAL ORDERING (I) *

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Abstract

Many types of nonlinear systems, which can be solved by ordered iterative methods, are discussed in unified form in the present paper. Under different initial conditions, some generalized ordered iterative methods are given, and the existence and uniqueness of the solution and the convergence of the methods are proved.

§ 1. Introduction

In this paper we consider nonlinear systems

$$\varphi(x)=x, \quad x\in R^n. \tag{1.1}$$

The partial ordering relation in R^n will be denoted, as usual, by "$\leqslant$", that is

$$x\leqslant y \leftrightarrow x_i\leqslant y_i \quad i=1, 2, \cdots, n,$$
$$x< y \leftrightarrow x_i< y_i \quad i=1, 2, \cdots, n.$$

For

$$\varphi(x)=g(x)+h(x)+c \tag{1.2}$$

where g, h: $R^n\to R^n$ are isotone and antitone mappings respectively, the problem has been solved quite satisfactorily. The purpose of this paper is to generalize the results in [1]—[4].

Suppose there are f_i: $R^{r_i}\times R^{s_i}\to R$, such that

$$\varphi_i(x)=f_i(A_ix, B_ix), \quad i=1, 2, \cdots, n \tag{1.3}$$

where $A_i\in R^{r_i\times n}$, $B_i\in R^{s_i\times n}$, $0\leqslant r_i$, $s_i\leqslant n$, $f_i(A_ix, B_iy)$ is isotone in x and antitone in y when they are comparable, that is, as $x\leqslant x'$, $y\geqslant y'$, $x\leqslant y$ or $y\leqslant x$, $x'\leqslant y'$ or $y'\leqslant x'$, we have

$$f_i(A_ix, B_iy)\leqslant f_i(A_ix', B_iy'), \quad i=1, 2, \cdots, n.$$

Example 1. For (1.2) we let

$$f(x, y)=g(x)+h(y)+c,$$
$$\varphi_i(x)=f_i(A_ix, B_ix)=g_i(A_ix)+h_i(B_ix)+c_i,$$

$A_i=B_i=I\in R^{n\times n}$, $i=1, 2, \cdots, n$.

Example 2[1]. For φ being diagonally isotone and off-diagonally antitone, we let

$$f_i(A_ix, B_iy)=\varphi_i(y+(x_i-y_i)e_i),$$
$$\varphi_i(x)=f_i(A_ix, B_ix),$$

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$A_i=e_i^T\in R^{1\times n}$, $B_i=[e_1, e_2, \cdots, e_{i-1}, e_{i+1}, \cdots, e_n]^T\in R^{(n-1)\times n}$, $i=1, 2, \cdots, n$, where $e_i=[0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0]^T\in R^n$.

Example 3[5]. $\varphi(x)=x-q'(x)^{-1}q(x)$, where q is order convex on a convex set $D\subseteq R^n$ i. e.

$$q(\lambda x+(1-\lambda)y)\leqslant\lambda q(x)+(1-\lambda)q(y)$$

whenever $x, y\in D$, $x\leqslant y$ or $y\leqslant x$ and $\lambda\in(0, 1)$. And if q is G-differentiable, $q'(x)\geqslant 0$, $q'(x)$ is isotone and $q(x)\geqslant 0$, then

$$\varphi_i(x)=f_i(A_ix), \quad i=1, 2, \cdots, n,$$

$A_i=I$, $s_i=0$. From

$$q'(\underline{x})(\bar{x}-\underline{x})\leqslant q(\bar{x})-q(\underline{x})\leqslant q'(\bar{x})(\bar{x}-\underline{x}), \quad \underline{x}\leqslant\bar{x}$$

we can prove that φ is isotone.

Most of the functions discussed in [1] (13.2—13.5) can be written in form of (1.3).

For simplicity, we suppose $A=A_i$, $B=B_i$, $i=1, 2, \cdots, n$, and consider

$$\varphi(x)=f(Ax, Bx)=x. \tag{1.4}$$

Clearly (1.4) is equivalent to (1.1). For other case, we can get similar results.

We define an n-dimensional interval vector

$$[\underline{x}, \bar{x}]=\{u\,|\,\underline{x}\leqslant u\leqslant\bar{x}\}$$

as an order interval, $\underline{x}, \bar{x}\in R^n$, and define.

$$N=\{1, 2, \cdots, n\},$$

$$[\underline{x}, \bar{x}]\subseteq[\underline{y}, \bar{y}]\leftrightarrow\underline{y}\leqslant\underline{x}\leqslant\bar{x}\leqslant\bar{y},$$

$$[\underline{x}, \bar{x}]_i\subset[\underline{y}, \bar{y}]_i\leftrightarrow\underline{y}_i\leqslant\underline{x}_i\leqslant\bar{x}_i\leqslant\bar{y}_i \quad\text{and}\quad \bar{y}_i-\underline{y}_i>\bar{x}_i-\underline{x}_i,$$

$$W[\underline{x}, \bar{x}]=(\bar{x}_1-\underline{x}_1, \cdots, \bar{x}_n-\underline{x}_n),$$

$$|x|=(|x_1|, |x_2|, \cdots, |x_n|)^T.$$

We will use the following lemmas.

Lemma 1. *Let $A\geqslant 0$ be an $n\times n$ matrix, and $\rho(A)$ be the spectral radius of the matrix A. Then*

(1) *A has a nonnegative real eigenvalue equal to its spectral radius.*

(2) *To $\rho(A)$, there is a corresponding eigenvector $x\geqslant 0$.*

(3) *$\rho(A)$ does not decrease when any entry of A is increased.*

(4) *$\alpha>\rho(A)$, if and only if $\alpha I-A$ is nonsingular and $(\alpha I-A)^{-1}\geqslant 0$.*

(5) *If A is an irreducible matrix, $\rho(A)$ increases when any entry of A increases.*

This lemma is a conclusion of the theorems about nonnegative matrices developed by Varge[6].

Lemma 2. *Let $B=I-w(\alpha I-A)$, $A\geqslant 0$, $\rho(A)<\alpha$, and $0<w\leqslant\min_i\{1/(1-a_{ii})\}$, $a_{ii}=e_i^TAe_i$. Then $B\geqslant 0$ and $\rho(B)<1$.*

Proof. From Lemma 1(4), $\alpha I-A$ is nonsingular and $(\alpha I-A)^{-1}\geqslant 0$. Therefore $\alpha-a_{ii}>0$, $i=1, 2, \cdots, n$, and $B=I-w(\alpha I-A)\geqslant 0$. Since $I-B=w(\alpha I-A)$, $(I-B)^{-1}\geqslant 0$. Let λ be an eigenvalue of B, and x be an eigenvector of B corresponding to the eigenvalue λ. Then

$$|\lambda|\,|x|=|\lambda x|=|Bx|\leqslant|B|\,|x|=B|x|$$