A SPECTRAL-DIFFERENCE METHOD FOR SOLVING TWO-DIMENSIONAL VORTICITY EQUATIONS*

GUO BEN-YU (郭本瑜)
(*Shanghai University of Science and Technology, Shanghai, China*)

Abstract

We construct a spectral-difference scheme for solving two-dimensional vorticity equation with a single periodical boundary condition. The conservation, the generalized stability and the convergence are proved. Both steady and unsteady problems are considered.

§ 1. Introduction

Let $\xi(x_1, x_2, t)$ and $\psi(x_1, x_2, t)$ be the vorticity function and the stream function respectively. ν is a positive constant. $f_j(x_1, x_2, t)$ and $\xi_0(x_1, x_2)$ are given. Let

$$I=\{x_2/0<x_2<2\pi\}, \quad Q=\{(x_1, x_2)/0<x_1<1, x_2\in I\}.$$

We consider the following two-dimensional vorticity equation

$$\begin{cases} \frac{\partial \xi}{\partial t}+\frac{\partial \psi}{\partial x_2}\frac{\partial \xi}{\partial x_1}-\frac{\partial \psi}{\partial x_1}\frac{\partial \xi}{\partial x_2}-\nu\left(\frac{\partial^2 \xi}{\partial x_1^2}+\frac{\partial^2 \xi}{\partial x_2^2}\right)=f_1, & \text{in } Q\times(0, T], \\ -\frac{\partial^2 \psi}{\partial x_1^2}-\frac{\partial^2 \psi}{\partial x_2^2}=\xi+f_2, & \text{in } Q\times[0, T], \\ \xi(x_1, x_2, 0)=\xi_0(x_1, x_2), & \text{in } \bar{Q}. \end{cases} \tag{1.1}$$

There is a lot of literature concerning the finite element methods and difference methods for solving (1.1); see, e.g., Raviart[1] and Guo Ben-yu[2,3]. But for any fixed scheme, the accuracy of the approximate solution is limited even if the solution of (1.1) is infinitely smooth.

In the past ten years, the spectral method for P. D. E. has developed rapidly, see Gottlieb, Orszag[4], Pasciak[5], Kreiss, Oliger[6] and Guo Ben-yu[7]. In particular, Guo Ben-yu[8] and Ma He-ping, Guo Ben-yu[9] proposed some spectral and pseudospectral schemes to solve (1.1). But all their works are for periodical problems.

In this paper, we assume that all functions are periodical only for the variable x_2 and thus we cannot use the full Fourier-spectral method. Such problems take place in the study of fluid flow in a tub. Following [10], we construct a class of spectral-difference scheme by using the Fourier-spectral method for the variable x_2 and the difference method for the variable x_1. If we choose the parameters in the scheme suitably, then the semi-discrete energy is kept unchanged. We strictly prove

*· Received September 25, 1986.

the generalized stability (see Guo Ben-yu[2, 11], and Griffiths[12]), from which the convergence follows with some assumption.

§ 2. The Scheme and the Conservations of the Approximate Solution

Suppose that all functions in (1.1) have the period 2π for the variable x_2.

Let h be the mesh spacing of x_1, $Mh=1$ and $I_h=\{x_1=jh/1\leqslant j\leqslant M-1\}$, $Q_h=I_h\times I$. Let τ be the mesh spacing of t, $S_\tau=\{t=k\tau/k=0, 1, 2, \cdots\}$. We define

$$u_{x_1}(x_1, x_2, t)=\frac{1}{h}(u(x_1+h, x_2, t)-u(x_1, x_2, t)),$$

$$u_{\bar{x}_1}(x_1, x_2, t)=u_{x_1}(x_1-h, x_2, t),$$

$$u_{\hat{x}_1}(x_1, x_2, t)=\frac{1}{2}(u_{x_1}(x_1, x_2, t)+u_{\bar{x}_1}(x_1, x_2, t)),$$

$$\Delta u(x_1, x_2, t)=u_{x_1,\bar{x}_1}(x_1, x_2, t)+\frac{\partial^2 u}{\partial x_2^2}(x_1, x_2, t),$$

$$u_t(x_1, x_2, t)=\frac{1}{\tau}(u(x_1, x_2, t+\tau)-u(x_1, x_2, t)).$$

The key problem for constructing a reasonable scheme is to simulate as many as possible the properties of the solution of (1.1). Indeed we have the following conservations

$$\begin{aligned}&\iint_Q \xi(x_1, x_2, t)dx_1dx_2\\&\quad+\int_0^t\left[\int_I\left\{\frac{\partial\psi}{\partial x_2}(1, x_2, y)\xi(1, x_2, y)-\frac{\partial\psi}{\partial x_2}(0, x_2, y)\xi(0, x_2, y)\right\}dx_2\right]dy\\&\quad-\nu\int_0^t\left[\int_I\left\{\frac{\partial\xi}{\partial x_1}(x_1, x_2, y)\Big|_{x_1=1}-\frac{\partial\xi}{\partial x_1}(x_1, x_2, y)\Big|_{x_1=0}\right\}dx_2\right]dy\\&=\iint_Q\xi_0(x_1, x_2)dx_1dx_2+\int_0^t\left[\iint_Q f_1(x_1, x_2, y)dx_1dx_2\right]dy\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}&\iint_Q \xi^2(x_1, x_2, t)dx_1dx_2\\&\quad+\int_0^t\left[\int_I\left\{\frac{\partial\psi}{\partial x_2}(1, x_2, y)\xi^2(1, x_2, y)-\frac{\partial\psi}{\partial x_2}(0, x_2, y)\xi^2(0, x_2, y)\right\}dx_2\right]dy\\&\quad+2\nu\int_0^t\left[\iint_Q\left\{\left(\frac{\partial\xi}{\partial x_1}(x_1, x_2, y)\right)^2+\left(\frac{\partial\xi}{\partial x_2}(x_1, x_2, y)\right)^2\right\}dx_1dx_2\right]dy\\&\quad-2\nu\int_0^t\left[\int_I\left\{\xi(x_1, x_2, y)\frac{\partial\xi}{\partial x_1}(x_1, x_2, y)\Big|_{x_1=1}\right.\right.\\&\quad\left.\left.-\xi(x_1, x_2, y)\frac{\partial\xi}{\partial x_1}(x_1, x_2, y)\Big|_{x_1=0}\right\}dx_2\right]dy\\&=\iint_Q\xi_0^2(x_1, x_2)dx_1dx_2+2\int_0^t\left[\iint_Q\xi(x_1, x_2, y)f_1(x_1, x_2, y)dx_1dx_2\right]dy.\end{aligned}\tag{2.2}$$