A MIXED STIFFNESS FINITE ELEMENT METHOD FOR NAVIER-STOKES EQUATION*

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Abstract

The paper is devoted to the study and analysis of the mixed stiffness finite element method for the Navier-Stokes equations, based on a formulation of velocity-pressure-stress deviatorics. The method used low order Lagrange elements, and leads to optimal error order of convergence for velocity, pressure, and stress deviatorics by means of the mesh-dependent norms defined in this paper. The main advantage of the MSFEM is that the streamfunction can not only be employed to satisfy the divergence constraint but stress deviatorics can also be eliminated at the element level so that it is unnecessary to solve a larger algebraic system containing stress multipliers, or to develop a special code for computing the MSFE solutions of the Navier-Stokes equations because we can use the computing codes used in solving the Navier-Stokes equations with the velocity-pressure formulation, or even the computing codes used in solving the problems of solid mechanics.

§1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^2$ with a sufficiently smooth boundary $\Gamma(=\partial\Omega)$. Then the Navier-Stokes equations governing the flow of the two-dimensional steady incompressible viscous fluid can be written as follows:

$$(u\cdot\nabla)u-\nu\Delta u+\nabla p=f, \quad \text{in } \Omega, \tag{1.1a}$$

$$\nabla\cdot u=0, \quad \text{in } \Omega, \tag{1.1b}$$

$$u=0, \quad \text{on } \Gamma, \tag{1.1c}$$

where $u=(u_1, u_2)$ are the velocities of flow, p is pressure, $f=(f_1, f_2)$ are body forces, and $\nu(>0)$ is the kinematic viscosity coefficient.

If the nonlinear convection terms in (1.1a) are cut out, then we obtain the so-called Stokes equations:

$$-\nu\Delta u+\nabla p=f, \quad \text{in } \Omega, \tag{1.2a}$$

$$\nabla\cdot u=0, \quad \text{in } \Omega, \tag{1.2b}$$

$$u=0, \quad \text{on } \Gamma. \tag{1.2c}$$

It is well known that considerable efforts have been made by both engineers and mathematicians concerning the construction of finite element solutions of the Stokes problem and the Navier-Stokes problem, see, e.g., [3, 6—13, 17—18, and 23]. What is worth mentioning is Zhou's paper, [23], which considers a new variational formulation of the MSFEM in [20] and [22] to avoid the trouble encountered in solving a larger system of equations owing to the additional viscous stress multiplier variables.

The purpose of this paper is to extend the results about the Stokes problem in

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[23] to the case of the Navier–Stokes problem. By treating the nonlinear terms with the upwind–diffusion scheme presented in [7], we prove the existence and uniqueness of the MSFE solutions of the Navier–Stokes problem, and obtain the optimal error estimates for velocity pressure and stress by virtue of special mesh–dependent norms. And L^2–estimates of velocity and pressure are also optimal. Moreover, stress multipliers can be eliminated at element level. An important step in practical computation is that velocity is first calculated in the divergencefree space, then pressure is found by the velocity obtained. To the author's knowledge, with the velocity–pressure–stress deviatorics formulation of the Navier–Stokes equations, the optimal error estimates in this paper are obtained for the first time.

An outline of the paper is as follows. The remaining part of the present section is to describe some definitions and symbols. Section 2 is devoted to the description of the MSFEM for the Navier–Stokes equations. In Section 3, we discuss the construction of the FE subspaces and prove their properties. Section 4 deals with the abstract results of the saddle–point problem. We get, in Sections 5 and 6, the error estimates of the solutions of the MSFEM for the Navier–Stokes problem in the sense of the mesh–dependent norms and L^2–norm.

Throughout this paper, we use the Sobolev space

$$H^m(\Omega)=\left\{v\in L^2(\Omega);\ \partial^\alpha v=\frac{\partial^{|\alpha|}v}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\in L^2(\Omega),\ |\alpha|=\alpha_1+\alpha_2\leqslant m\right\}$$

equipped with the following norm and semi–norm:

$$\|v\|_{m,\Omega}=\{\sum_{|\alpha|\leqslant m}\|\partial^\alpha v\|^2_{0,\Omega}\}^{1/2},$$

$$|v|_{m,\Omega}=\{\sum_{|\alpha|=m}|\partial^\alpha v|^2_{0,\Omega}\}^{1/2},$$

where m (>0) is an integer. We denote by $H^{1/2}(\Gamma)$ the trace space which consists of functions defined on the boundary Γ. Moreover, some special spaces will be defined when they appear. As to the details of Sobolev spaces, see [1, 8], and [13—14].

For convenience, we do not make distinction between the vector–valued function and the scalar–valued function. The standard summation convention is empolyed. We denote by n and t the unit normal and tangent vector on some boundary respectively. And c stands for the generic constant unless particular explanation is given.

§ 2. The MSFE Formulation

To facilitate the analysis below, we introduce the following relations:

$$\mu=2\nu,$$

$$\varepsilon(v)=\{\varepsilon_{ij}(v)\}_{1\leqslant i,j\leqslant 2},$$

$$\varepsilon_{ij}(v)=(\partial_i v_j+\partial_j v_i)/2,\quad 1\leqslant i,\ j\leqslant 2.$$

Then the Navier–Stokes equations (1.1) can be rewritten as

$$\sigma=\mu\varepsilon(u),\quad \text{in } \Omega, \tag{2.1a}$$

$$\nabla\cdot u=0,\quad \text{in } \Omega, \tag{2.1b}$$

$$(u\cdot\nabla)u-\nabla\cdot\sigma+\nabla p=f,\quad \text{in } \Omega, \tag{2.1c}$$