# ACCELERATION OF THE CONVERGENCE IN FINITE DIFFERENCE METHOD BY PREDICTOR-CORRECTOR AND SPLITTING EXTRAPOLATION METHODS*

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## Abstract

Two types of combination methods for accelerating the convergence of the finite difference method are presented. The first is based on an interpolation principle (correction method) and the second one on extrapolation principle. They improve the convergence from $O(h^2)$ to $O(h^4)$. The main advantage, when compared with standard methods, is that the computational work can be splitted into independent parts, which can then be carried out in parallel.

## § 1. Introduction

We study the finite difference approximation of the solution, $u$, to the model problem

$$\begin{cases} \Delta u=f, & \text{in } \Omega, \\ u=g, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

in the 2 or 3-dimensional domain $\Omega$ with boundary $\partial\Omega$. Suppose that $\Omega$ consists of some squares in the 2-dimensional case and of some cubes in the 3-dimensional case. Furthermore, suppose that the solution of (1.1) is smooth enough.

Let $u_h$ be the solution of the approximate finite difference analogy to (1.1) with mesh size $h$:

$$\begin{cases} \Delta_h u_h=f, & \text{in } \Omega_h^d,\ d=2,3, \\ u_h=g, & \text{on } \partial\Omega_h^d. \end{cases} \tag{1.2}$$

Here $\Delta_h$ denotes the 5-point approximation of the Laplace operator $\Delta$ in the 2-dimensional case, and the 7-point approximation in the 3-dimensional case as usual.

It is well known that

$$u-u_h=h^2e+O(h^4) \text{ in } \Omega_h^d, \text{ for } h\to 0, \tag{1.3}$$

where $e$ is the solution of a correction differential equation independent of $h$ ([3, 7]). Function $e$ can be estimated as

$$-h^2e=\frac{4}{3}(u_h-u_{h/2})+O(h^4). \tag{1.4}$$

The more accurate solution, $u_{h/2}$, may be corrected by $\frac{1}{3}(u_{h/2}-u_h)$, and the correction taken as an error estimate. The disadvantage of the above extrapolation procedure is the computation of another solution with a smaller parameter $(h/2)$, which involves solving once again a finite difference equation of much larger size than the one corresponding to the original $h$ for multidimensional problems.

In this paper two methods are presented which lead to accuracy $O(h^4)$ but which are of a lower computational complexity than the standard extrapolation method. They are especially efficient when parallel architecture of the computer system is used. The methods consist of a predictor-corrector type method, and a splitting extrapolation method. The methods will be introduced in sections 2 and 3. Finally, in section 4 we give some results of numerical tests. In the two-dimensional case we have compared the presented correction and splitting extrapolation methods with the standard 5-point scheme and multigrid method; in the three-dimensional case we have compared our method with 9-point and 15-point schemes, and with the standard extrapolation method. All computations have been carried out with the conventional computer system. Consequently, the parallelization properties of the correction method and the splitting extrapolation method have not been utilized. In spite of that, the methods presented here seem to be superior compared with standard ones.

For a survey of extrapolation methods in FE and FD-schemes we refer to survey work [6]. Especially, see [1] for FEM and [11] for FD in connection with multigrid method.

## § 2. Predictor-Corrector Type Procedure

Define the (uniform regular) lattice domain

$$\Omega_h^d \equiv \Omega(h, \cdots, h)$$
$$= \{(x_1, \cdots, x_d) \in \mathbb{R}^d \mid x_i = m_i h,\ m_i = 0, \pm 1, \cdots, \pm n,\ i = 1, \cdots, d,\ nh = 1\}.$$

Let operators $\Delta_h^+$ and $\Delta_h^\times$ be approximations for the Laplace operator $L = \Delta$ at point $x \in \operatorname{int} \Omega_h^d$ (where $\operatorname{int} \Omega$ denotes the interior of $\Omega$). For $d = 2$, $\Delta_h^+$ is the 5-point difference operator, where

$$\Delta_h^+ u(x_1, x_2) = L_5^h u(x_1, x_2) = \frac{1}{h^2}\{u(x_1 - h, x_2) + u(x_1 + h, x_2)$$
$$+ u(x_1, x_2 - h) + u(x_1, x_2 + h) - 4u(x_1, x_2)\},$$

and $\Delta_h^\times$ the 5-point difference operator, where

$$\Delta_h^\times u(x_1, x_2) = L_5^h u(x_1, x_2) = \frac{1}{2h^2}\{u(x_1 - h, x_2 - h) + u(x_1 + h_1, x_2 + h)$$
$$+ u(x_1 - h, x_2 + h) + u(x_1 + h, x_2 - h) - 4u(x_1, x_2)\}.$$

For $d = 3$, $\Delta_h^+$ is the 7-point difference operator, and

$$\Delta_h^+ u(x_1, x_2, x_3) = L_7^h u(x_1, x_2, x_3)$$
$$= \frac{1}{h^2}\{u(x_1 - h, x_2, x_3) + u(x_1 + h, x_2, x_3) + u(x_1, x_2 - h, x_3)$$
$$+ u(x_1, x_2 + h, x_3) + u(x_1, x_2, x_3 - h)$$
$$+ u(x_1, x_2, x_3 + h) - 6u(x_1, x_2, x_3)\},$$

and with $\Delta_h^\times$ as the 9-point difference operator, we have