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ON A ONE-DIMENSIONAL DIFFERENCE SCHEME IN REACTION DIFFUSION*

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I. Introduction

Ludwig, Jones and Holling (1978) modelled the spruce budworm problem by using a scaled ordinary differential equation. Spatial effects were introduced by Ludwig, Aronson, and Weinberger (1979) by the addition of a diffusion term to the equation. Recently Guo Ben-yu et al. (1983) obtained some precise results for the bifurcation lengths in circular and rectangular regions. These analytic results are extended in the present paper to cover the case of difference equations in reaction diffusion. The analysis is restricted to one space dimension and only the linear and nonlinear logistic models are considered. Despite these restrictions, the techniques used and the comparison principles proved are useful for more general problems.

II. The Linear Model

In this section we consider the linear model of the spruce budworm problem. Let the region considered be an infinite strip of breadth l and $W(y, t)$ be the scaled density of the budworm population where (see Ludwig, Aronson and Weinberger, 1979)

$$\begin{cases} \frac{\partial W}{\partial t}-\frac{\partial^2 W}{\partial y^2}-W=0, & 0<y<l,\ t>0, \\ W(0, t)=W(l, t)=0, & t\geqslant 0, \\ W(y, 0)=W_0(y), & 0<y<l, \end{cases} \tag{1}$$

where $0\leqslant U_0(x)\leqslant M_0$. Let $y=lx$, $U(x, t)=W\left(\frac{y}{l}, t\right)$ and $\varepsilon=\frac{1}{l^2}$. Then (1) becomes

$$\begin{cases} \frac{\partial U}{\partial t}-\varepsilon\frac{\partial^2 U}{\partial x^2}-U=0, & 0<x<1,\ t>0, \\ U(0, t)=U(1, t)=0, & t\geqslant 0, \\ U(x, 0)=U_0(x), & 0<x<1. \end{cases} \tag{2}$$

We cover the region $[0\leqslant x\leqslant 1]\times[t\geqslant 0]$ by a rectangular grid, where h and τ are the mesh sizes of the variables x and t respectively; also, $Nh=1$ where N is an integer. We define

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$$I_h=\{x\,|\,x=h,\ 2h,\ \cdots,\ (N-1)h\},\quad \bar I_h=I_h+\{0\}+\{1\}.$$

Let $\eta^k(x)$ be the value of the mesh function η at point $x\in\bar I_h$, $t=k\tau$ $(k\geqslant 0)$. We use the following notations

$$\eta_x^k(x)=\frac{1}{h}[\eta^k(x+h)-\eta^k(x)],\qquad \eta_{\bar x}^k(x)=\eta_x^k(x-h),$$

$$\eta_{x\bar x}^k(x)=\frac{1}{h^2}[\eta^k(x+h)-2\eta^k(x)+\eta^k(x-h)],$$

and

$$\eta_t^k(x)=\frac{1}{\tau}[\eta^{k+1}(x)-\eta^k(x)].$$

We introduce the discrete scalar product and the norms as follows:

$$(\eta^k,\ \xi^k)=h\sum_{x\in I_h}\eta^k(x)\cdot\xi^k(x),$$

$$\|\eta^k\|^2=(\eta^k,\ \eta^k),\qquad |\eta^k|_1^2=\frac{1}{2}\|\eta_x^k\|^2+\frac{1}{2}\|\eta_{\bar x}^k\|^2,\qquad \|\eta^k\|_\infty=\max_{x\in I_h}|\eta^k(x)|.$$

It is clear that

$$-(\eta^k,\ \eta_{x\bar x}^k)=|\eta^k|_1^2+\frac{1}{2h}[\eta(h)]^2+\frac{1}{2h}[\eta(1-h)]^2. \tag{3}$$

Let $u^k(x)$ be the approximation to $U^k(x)$. The Crank-Nicolson scheme for solving (2) is

$$\begin{cases}u_t^k(x)-\dfrac{\varepsilon}{2}(u_{x\bar x}^k(x)+u_{x\bar x}^{k+1}(x))-\dfrac{1}{2}[u^k(x)+u^{k+1}(x)]=0, & x\in I_h,\ k\geqslant 0,\\ u^k(0)=u^k(1)=0, & k\geqslant 0,\\ u^0(x)=U_0(x), & x\in I_h.\end{cases} \tag{4}$$

Let

$$u^k(x)=\sum_{\beta=1}^{N-1}a_\beta b^k(\beta)\sin\beta\pi x,\quad x\in\bar I_h,\ k\geqslant 0,$$

where

$$U_0(x)=\sum_{\beta=1}^{N-1}a_\beta\sin\beta\pi x,\quad x\in\bar I_h.$$

Then

$$b(\beta)=\frac{1-\dfrac{2\varepsilon\tau}{h^2}\sin^2\dfrac{\beta\pi h}{2}+\dfrac{\tau}{2}}{1+\dfrac{2\varepsilon\tau}{h^2}\sin^2\dfrac{\beta\pi h}{2}-\dfrac{\tau}{2}}.$$

We define

$$\varepsilon_h^*=\frac{h^2}{4\sin^2\dfrac{\pi h}{2}}$$

and let $w^k(x)$ be the solution of (4) with $w^0(x)\equiv M_0$. Then

$$w^k(x)=\frac{2M_0}{N}\sum_{\beta=1}^{\frac{N}{2}}\frac{b^k(\beta)\cos\dfrac{\beta\pi}{2N}\sin(2\beta-1)\pi x}{\sin\dfrac{\beta\pi}{2N}}.$$

Clearly $w^k(x)\leqslant w^k\left(\frac{1}{2}\right)$ and $w^k\left(\frac{1}{2}\right)\to 0$ as $k\to\infty$ provided $\varepsilon>\varepsilon_h^*$.

Now suppose

$$\tau\leqslant\min\left(2,\ \frac{2h^2}{2\varepsilon-h^2}\right).$$

By Lemma A_2 (see Appendix), $0\leqslant u^k(x)\leqslant w^k(x)$ and thus $u^k(x)\to 0$ as $k\to\infty$ as long