ALGEBRAIC CRITERION OF CONSISTENCY FOR GENERAL LINEAR METHODS OF ORDINARY DIFFERENTIAL EQUATIONS*

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§ 1. Introduction

Consider the initial value problem of ordinary differential equations

$$\begin{cases} \dfrac{dy}{dx}=f(y), \\ y(x_0)=y_0, \end{cases} \tag{1}$$

where $x\in R$, $y\in R^m$. In unifying the theory of various linear methods, Burrage and Butcher [1] presented the following general linear methods:

$$\begin{cases} Y_i^{(n)}=h\sum_{j=1}^{s} c_{ij}^{11}f(Y_j^{(n)})+\sum_{j=1}^{k} c_{ij}^{12}y_j^{(n-1)}, & i=1, 2, \cdots, s, \\ y_i^{(n)}=h\sum_{j=1}^{s} c_{ij}^{21}f(Y_j^{(n)})+\sum_{j=1}^{k} c_{ij}^{22}y_j^{(n-1)}, & i=1, 2, \cdots, k. \end{cases} \tag{2}$$

There are s internal vectors $Y_1^{(n)}, \cdots, Y_s^{(n)}$ and k external vectors $y_1^{(n)}, \cdots, y_k^{(n)}$, respectively. For the matrix-vector form, set

$$Y^{(n)}=Y_1^{(n)}\oplus Y_2^{(n)}\oplus\cdots\oplus Y_s^{(n)},$$
$$y^{(n)}=y_1^{(n)}\oplus y_2^{(n)}\oplus\cdots\oplus y_k^{(n)},$$
$$F(Y^{(n)})=f(Y_1^{(n)})\oplus f(Y_2^{(n)})\oplus\cdots\oplus f(Y_s^{(n)});$$
$$C_{11}=(c_{ij}^{11})_{s\times s},\ C_{12}=(c_{ij}^{12})_{s\times k},\ C_{21}=(c_{ij}^{21})_{k\times s},\ C_{22}=(c_{ij}^{22})_{k\times k};$$
$$C=\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},\quad [C]=\begin{pmatrix} C_{11}\otimes I_s & C_{12}\otimes I_k \\ C_{21}\otimes I_s & C_{22}\otimes I_k \end{pmatrix},$$

where I's are unit matrices. By means of the partitioned matrix C, the general linear methods (2) are expressed as

$$\begin{pmatrix} Y^{(n)} \\ y^{(n)} \end{pmatrix}=[C]\begin{pmatrix} hF(Y^{(n)}) \\ y^{(n-1)} \end{pmatrix}. \tag{3}$$

Burrage and Butcher [1] defined that if there exist vectors $u, v\in R^k$ such that

$$C_{12}u=e,\ C_{22}u=u, \tag{4}$$
$$C_{21}e+C_{22}v=u+v, \tag{5}$$

then the method (3) is said to be consistent, where $e=(1\ 1\ \cdots\ 1)^T$.

There are two questions: First, three submatrices C_{12}, C_{21} and C_{22} are constrained in the consistency conditions (4) and (5); and what about the submatrix

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C_{11}? Second, when the method (3) is given, how to determine the vectors u and v, and how to discuss the consistency of order $p\geqslant 1$?

Frank, Schneid and Ueberhuber [2] defined the so called B-consistency for the general linear methods. The method (3) is B-consistent of order $p\geqslant 1$, if the numerical solution ξ obtained one step forward from $y(x)$ (steplength h) and the exact solution $y(x+h)$ of problem (1) satisfy

$$\|\xi-y(x+h)\|\leqslant Dh^{p+1}, \quad h\in(0, h_0], \tag{6}$$

where D and h_0 depend only on the logarithmic norm of $\frac{\partial f}{\partial y}$ and other derivatives of f.

This consistency condition seems to be strengthened in order to gain the B-convergence. And no algebraic criterion of the consistency is given.

In this paper, the necessary and sufficient conditions of consistency of order $p\geqslant 1$ for the general linear method (3) are presented. They have very wide application in various linear methods.

§ 2. Main Result

According to the feature of general linear method (3), the internal vectors $Y_i^{(n)}$ must be in the domain of f, so they usually are approximations of the exact solution $y(x)$ of problem (1) at the point x_0+nh+w_ih:

$$Y_i^{(n)}\sim\eta_i^{(n)}=y(x_0+nh+w_ih), \quad i=1, 2, \cdots, s. \tag{7}$$

The external vectors $y_i^{(n)}$ are relatively free; they may be expressed in a rather general form:

$$y_i^{(n)}\sim\zeta_i^{(n)}=\alpha_iy(x_0+nh+\gamma_ih)+\beta_ihy'(x_0+nh+\delta_ih), \quad i=1, 2, \cdots, k. \tag{8}$$

When the method (3) is given, the parameters w_i, α_i, β_i, γ_i and δ_i can be determined. Particularly, $\alpha_i^2+\beta_i^2\neq 0$ and $\alpha_i\cdot\beta_i=0$. Set

$$H^{(n)}=\eta_1^{(n)}\oplus\eta_2^{(n)}\oplus\cdots\oplus\eta_s^{(n)}, \; Z^{(n)}=\zeta_1^{(n)}\oplus\zeta_2^{(n)}\oplus\cdots\oplus\zeta_k^{(n)}. \tag{9}$$

As usual,

$$\zeta_i^{(n-1)}=\alpha_iy(x_0+(n-1)h+\gamma_ih)+\beta_ihy'(x_0+(n-1)h+\delta_ih), \quad i=1, 2, \cdots, k. \tag{10}$$

Definition. *For the general linear method (3), if*

$$\left\|\begin{pmatrix}H^{(n)}\\ Z^{(n)}\end{pmatrix}-[C]\begin{pmatrix}hF(H^{(n)})\\ Z^{(n-1)}\end{pmatrix}\right\|=O(h^{p+1}), \tag{11}$$

then it is said to be consistent of order p.

Obviously, this is a measure about the local discretization error of the scheme (3), and it coincides with the proper sense of the concept "consistency". The key to the establishment of consistency definition (11) lies in expressions (7) and (8) of the internal and external vectors.

Before formulating the algebraic criterion of consistency for general linear methods, we set the following symbols.

For the vectors