AN INTERIOR POINT METHOD FOR LINEAR PROGRAMMING*[1)]

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Abstract

In this paper we present an interior point method which solves a linear programming problem by using an affine transformation. We prove under certain assumptions that the algorithm converges to an optimal solution even if the dual problem is degenerate as long as the primal is bounded, or to a ray direction if the optimal value of the objective function is unbounded.

§1. Introduction

This paper presents an interior point method for solving a linear programming problem (LP), and proves the convergence of the algorithm under certain assumptions. The algorithm was previously mentioned in [7], in which the convergence was proven under the assumptions that the primal and dual problems are both nondegenerate, and that the problem is bounded. In fact, the algorithm performs well in more general cases, but under an additional assumption that the LP problem satisfies the condition F, which will be defined in Section 2. The condition F implies that if the feasible region of the LP problem is bounded, then the primal problem is nondegenerate and that if the feasible region is unbounded, then the primal problem is nondegenerate, and there are at least $m+1$ nonzero components in the vector of any ray direction.

In this paper we show, under the assumption that the LP problem satisfies the above condition F, that the algorithm converges to an optimal solution for a bounded primal problem, even if the dual problem is degenerate, and to an extreme point if the dual problem is nondegenerate. We also prove that in the case of the unboundedness of the LP problem the algorithm converges to a ray direction, along which the minimum value is unbounded.

§2. Algorithm

This section describes the interior point method for solving the LP problem by use of affine transformations.

We consider the following standard form of the linear programming problem:

$$\text{minimize } z = c^T x, \tag{2.1}$$

$$\text{subject to } Ax = b, \tag{2.2}$$

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1) This work was done while the author visited the Dept. of Industrial Engineering and Operations Research, University of California, Berkeley, USA.

$$x \geqslant 0, \tag{2.3}$$

where A is an $m \times n$ real matrix with rank m, and $m < n$. b and c are real vectors in R^m and R^n, respectively, and x is a real variable in R^n.

Let $S = [x\colon Ax = b,\ x \geqslant 0]$ denote the feasible region of the LP problem, then the LP problem is feasible if S is nonempty.

Definition 2.1. *The LP problem satisfies the condition F if S is nonemptg, and for any $x \in S$, the matrix AD^2A^T is of full rank, where D is a diagonal matrix containing components of x.*

From the definition, it is easy to see that if the feasible region S is bounded, then the condition F is equivalent to nondegeneracy of the primal, and if S is unbounded, then for any ray direction ν of S, the matrix $AD_\nu^2A^T$ is of full rank, where D_ν is a diagonal matrix containing components of ν.

Suppose that the LP problem satisfies the condition F, and that $\bar{x}$ is a strictly interior feasible point. Then an affine transformation and its inverse can be defined as follows:

$$x' = D^{-1}x, \tag{2.4}$$

$$x = Dx'. \tag{2.5}$$

With the above transformation the original LP problem is transformed into the following linear programming problem in x'-space:

$$\text{minimize } z = c^T Dx', \tag{2.6}$$

$$\text{subject to } ADx' = b, \tag{2.7}$$

$$x' \geqslant 0, \tag{2.8}$$

where D is a diagonal matrix containing components of $\bar{x}$.

Obviously, from the definition of D the point $\bar{x}$ in x-space is mapped into the point $e^T = (1, 1, \cdots, 1)$ in x'-space. Hence, in order to get the maximum rate of decrease of the objective function, a large step away from the point e to a new point in x'-space is taken along the negative of the projective gradient direction. The new point is transformed back into the x-space, and an iterative point is obtained.

Given a strictly interior point $x^{(0)}$, the algorithm described above, which creates a sequence of points $x^{(0)}, x^{(1)}, \cdots$, is defined more formally as follows.

Algorithm A.

$k\colon = 0$. Given $x^{(0)}$, a strictly interior feasible point.

(1) Define

$$D_k = \operatorname{diag}(x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)}), \tag{2.9}$$

$$A_k = AD_k. \tag{2.10}$$

(2) Compute the vector $c_p^{(k)}$ and its norm $\|c_p^{(k)}\|_2$ by

$$c_p^{(k)} = [I - A_k^T (A_k A_k^T)^{-1} A_k] D_k c \tag{2.11}$$

and

$$\|c_p^{(k)}\|_2 = \sqrt{(c_p^{(k)})^T (c_p^{(k)})}. \tag{2.12}$$

(3) Normalize $c_p^{(k)}$,