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SOLUTION OF THE TWO-DIMENSIONAL STEFAN PROBLEM BY THE SINGULARITY-SEPARATING METHOD*1)

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Abstract

In this paper the idea of "singularity-separating" presented in [10] is used to solve a two-dimensional phase-change problem. A difference scheme with second-order accuracy everywhere, including the region near the boundary between two phases, is constructed for the above problem. Through the computation it is shown that the singularity-separating method, whose accuracy is high, is efficient for two-dimensional phase-change problem.

I. Introduction

The Stefan problem, a moving boundary problem for parabolic partial differential equations, is an important subject studied by many scholars for years. It is often met with in engineerings and geophysics. For the multi-dimensional Stefan problem the analytic solution cannot be found except for only a few special cases, and therefore people devote themseft to finding its numerical solution. At present the difference methods and the finite element methods are the main methods for this problem[1-9]. Besides, there is a method in which the original equation is transformed to a new equation by using the internal energy function, and then the difference equations are obtained from the new equation. In the Stefan problem the boundaries among the media with different phases move with time t, and so are called moving boundaries. On the moving boundaries the solutions are weakly discontinuous and there exist exothermic processes or endothermic processes. Such a singularity makes it very difficult to find a numerical method with high accuracy for this problem.

We have presented a new numerical method, the singularity-separating method for the Stefan problem——a heat conduction problem with phase change. Its main idea goes as follows: First a curvilinear coordinate transformation is used to turn the moving phase-change boundaries into fixed boundaries of straight lines under the new coordinates. Thus the whole region in the new coordinates is divided into several rectangular subregions by the phase-change boundaries. Then in the subregions a stable difference scheme is constructed for the heat conduction equation

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under the new coordinates. Since difference equations are constructed in each subregion, there is no difference across discontinuities in the difference equations. Finally a simultaneous system composed of the difference equations in these subregions and the Stefan condition can be solved in order to obtain the solution.

The problem in one dimension has been discussed in [6]. Here we study the case in two dimensions.

II. Mathematical Formulation of the Problem

The problem with phase change is studied in the region $D=\{(x, y)\,|\,0<x<X, 0<y<H\}$. The solid phase region is denoted as $\Omega_1(t)$ and the liquid phase region as $\Omega_2(t)$. Suppose that there is only one phase-change boundary, which is denoted as $\Gamma(t)$ (see Fig. 1).

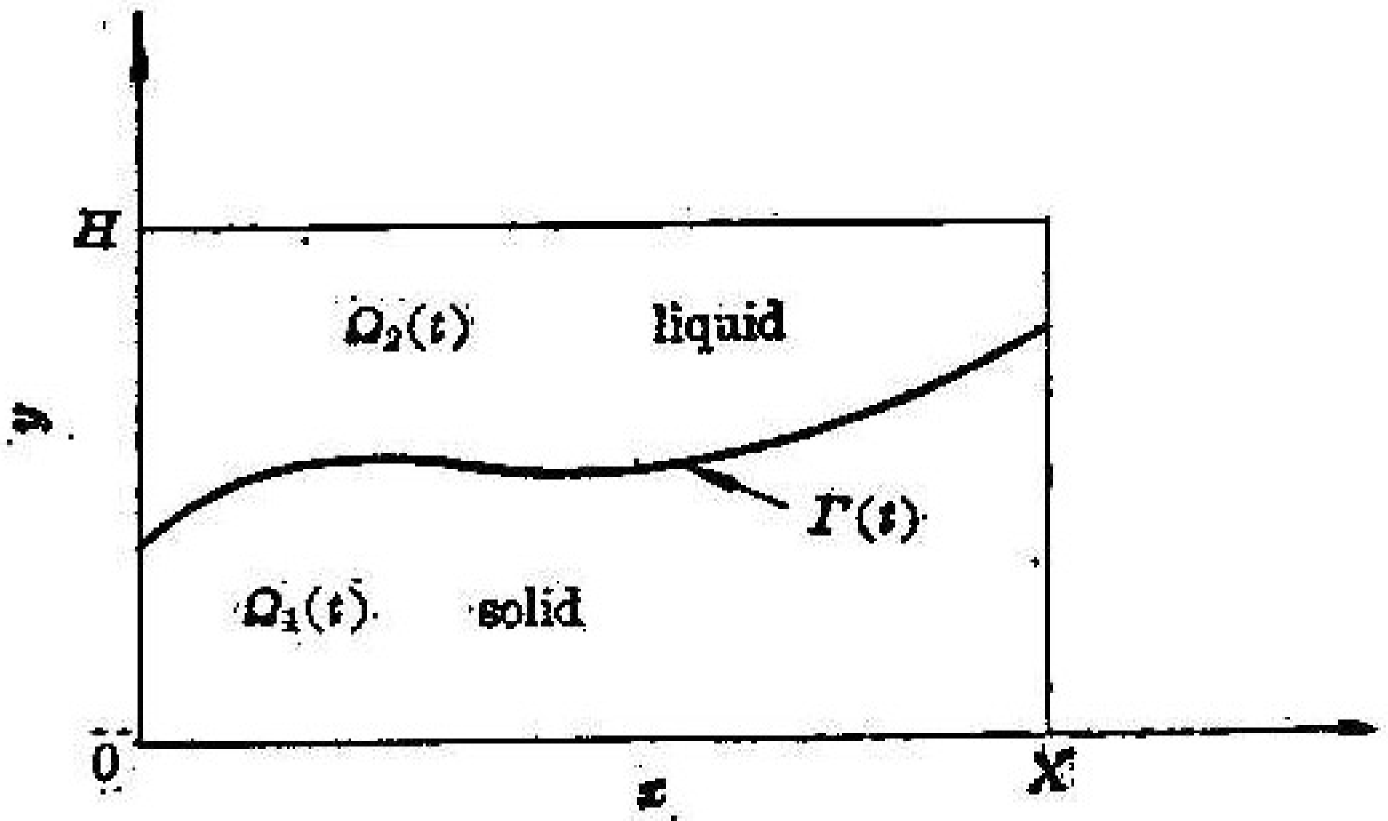

Fig. 1

According to the heat conservation law, the heat conduction problems in two dimensions in the solid region and the liquid region can be respectively described using the following formulae

$$C_1\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[k_1\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[k_1\frac{\partial u}{\partial y}\right],\quad 0<x<X,\ 0<y<f(x, t),\ t>0; \tag{2.1}$$

$$C_2\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[k_2\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[k_2\frac{\partial u}{\partial y}\right],\quad 0<x<X,\ f(x, t)<y<H,\ t>0. \tag{2.2}$$

Here $C_i=C_i(u)$, $i=1, 2$, stand for the specific thermal capacities (that is, the quantity of heat which per volume of substance needs for its temperature to increase by 1°C); $k_i=k_i(u)$, $i=1, 2$, for the coefficients of heat conduction. The subscripts 1 and 2 stand for the solid region and the liquid region respectively.

Suppose that the equation of the phase-change boundary $\Gamma(t)$ is $y=f(x, t)$. On the surface $y=f(x, t)$, the connective condition can be written as

$$u^-(x, f(x, t), t)=u^+(x, f(x, t), t)=u_f; \tag{2.3}$$

and

$$\lambda\frac{d\xi_n}{dt}=\left(k_1\frac{\partial u^-}{\partial n}-k_2\frac{\partial u^+}{\partial n}\right)\Big|_{\Gamma(t)}. \tag{2.4}$$

Here u^- and u^+ respectively represent the values of u on the lower side and on the upper side of $y=f(x, t)$, $\boldsymbol{n}$ stands for the unit normal, $d\xi_n$ for the variation of distance along $\boldsymbol{n}$ (see Fig. 2), λ for the latent heat of phase-change and u_f for the phase-change temperature.

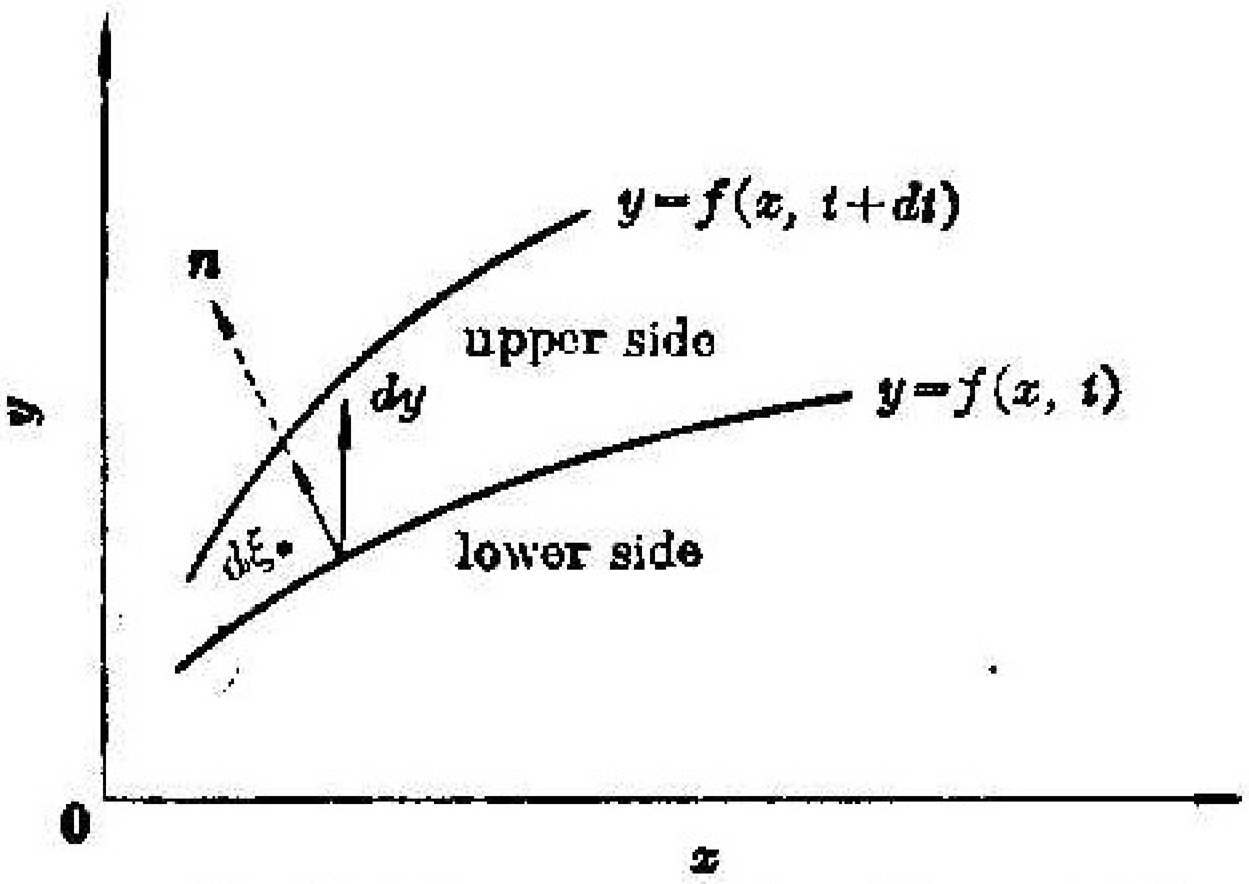

Fig. 2 The normal direction and $d\xi_n$

Formula (2.4) is called the Stefan condition, and can also be written as

$$\lambda\frac{\partial f}{\partial t}=k_1\frac{\partial u^-}{\partial y}-k_2\frac{\partial u^+}{\partial y}-\frac{\partial f}{\partial x}\times\left(k_1\frac{\partial u^-}{\partial x}-k_2\frac{\partial u^+}{\partial x}\right),\quad \text{on } y=f(x, t).$$