ESTIMATION OF THE SEPARATION OF TWO MATRICES (II)*

SUN JI-GUANG (孙继广)
(*Computing Center, Academia Sinica, Beijing, China*)

Abstract

In this paper we give a lower bound of the separation $\mathrm{sep}_F(A, B)$ of two diagonalizable matrices A and B. The key to finding the lower bound of $\mathrm{sep}_F(A, B)$ is to find an upper bound for the condition number $\varkappa(Q)$ of a transformation matrix Q which transforms a diagonalizable matrix A to a diagonal form. The obtained lower bound of $\mathrm{sep}_F(A, B)$ involves the eigenvalues of A and B as well as the departures from normality $\Delta_F(A)$ and $\Delta_F(B)$.

This is a continuation of [6]. In addition to the notation explained in [6] we use $\mathbb{C}^n$ for the n-dimensional column vector space, and $\mathfrak{R}(X)$ for the column space of a matrix X. $\oplus$ denotes the direct sum of subspaces, and $\mathscr{X}^{\perp}$ the orthogonal complement of a subspace $\mathscr{X}$. Besides, X^H stand for conjugate transpose of X.

§ 4. An Upper Bound for the Spectral Condition Number of a Diagonalizable Matrix

Let A and B be diagonalizable matrices with the eigenvalues $\{\lambda_i\}$ and $\{\mu_i\}$ respectively, Q_A and Q_B be transformation matrices which transform A and B to diagonal forms. It is proved that if we set

$$\delta(A, B)=\min_{i,j}|\lambda_i-\mu_j| \tag{4.1}$$

and

$$\varkappa(Q)=\|Q\|_2\|Q^{-1}\|_2, \tag{4.2}$$

then[5, 6]

$$\frac{\delta(A, B)}{\varkappa(Q_A)\varkappa(Q_B)}\leqslant \mathrm{sep}_F(A, B)\leqslant\delta(A, B). \tag{4.3}$$

Therefore, estimation of a lower bound for the separation $\mathrm{sep}_F(A, B)$ is reduced to estimations of upper bounds for the condition numbers $\varkappa(Q_A)$ and $\varkappa(Q_B)$.

In this section we use the characteristic of a diagonalizable matrix A to give an upper bound for the spectral condition number $\inf\limits_Q \varkappa(Q)$ of A, here the inf taking over all Q which similarity transforms A to a diagonal form.

For a nonsingular matrix Q, we set

$$K(Q)=\|Q\|_F\|Q^{-1}\|_F. \tag{4.4}$$

The following lemma delineates the relation between the $K(Q)$ and $\varkappa(Q)$.

Lemma 4.1. *Suppose that $Q\in\mathbb{C}^{m\times m}$ is nonsingular. Then*

* Received February 17, 1984.

$$1+\frac{K(Q)-m+\sqrt{K^2(Q)-m^2}}{m}\leqslant\varkappa(Q)$$

$$\leqslant 1+\frac{K(Q)-m+\sqrt{[K(Q)-m+2]^2-4}}{2}. \tag{4.5}$$

Proof. Let $K=K(Q)$ and $\varkappa=\varkappa(Q)$. By Theorem 1 of [4],

$$m-2+\varkappa+\varkappa^{-1}\leqslant K\leqslant\frac{1}{2}m(\varkappa+\varkappa^{-1}). \tag{4.6}$$

Combining $\varkappa+\varkappa^{-1}\geqslant 2$ and the first inequality of (4.6), we get $K\geqslant m$. From the second inequality of (4.6),

$$0<\varkappa\leqslant 1-\frac{\sqrt{K^2-m^2}-(K-m)}{m}, \quad \varkappa\geqslant 1+\frac{K-m+\sqrt{K^2-m^2}}{m}; \tag{4.7}$$

and from the first inequality of (4.6),

$$1-\frac{\sqrt{(K-m+2)^2-4}-(K-m)}{2}\leqslant\varkappa\leqslant 1+\frac{K-m+\sqrt{(K-m+2)^2-4}}{2}. \tag{4.8}$$

Observe that

$$\frac{K-m+\sqrt{K^2-m^2}}{m}\leqslant\frac{K-m+\sqrt{(K-m+2)^2-4}}{2},$$

$$0\leqslant\frac{\sqrt{K^2-m^2}-(K-m)}{m}\leqslant\frac{\sqrt{(K-m+2)^2-4}-(K-m)}{2}$$

and

$$\frac{\sqrt{K^2-m^2}-(K-m)}{m}=0 \quad \text{iff } K=m \quad \text{iff } \varkappa=1,$$

hence, from (4.7) and (4.8) we obtain the inequalities (4.5) at once. ∎

Now we cite a theorem proved by Elsner[3], which is a generalization of a result due to Smith[4].

Theorem 4.1. *Suppose that* $A\in\mathbb{C}^{m\times m}$ *with different eigenvalues* $\lambda_1, \lambda_2, \cdots, \lambda_r$ *of multiplicities* $m_1, m_2, \cdots, m_r$ *respectively. Let* $\mathbb{C}^m=\mathscr{X}_1\oplus\mathscr{X}_2\oplus\cdots\oplus\mathscr{X}_r$, $\mathscr{X}_i$ *be the invariant subspace of A corresponding to the* λ_i *with* $\dim(\mathscr{X}_i)=m_i$, $i=1, 2, \cdots, r$. *If we set* $\mathscr{Y}_i=\bigcap_{j\neq i}\mathscr{X}_j^{\perp}$, $i=1, 2, \cdots, r$, *and*

$$\mathscr{Q}=\{Q=(Q_1, Q_2, \cdots, Q_r):\Re(Q_i)=\mathscr{X}_i, \ i=1, \cdots, r\},$$

then

$$\min_{Q\in\mathscr{Q}} K(Q)=\sum_{i=1}^{r}\sum_{j=1}^{m_i}\frac{1}{\sigma_i^{(j)}}, \tag{4.9}$$

where $\{\sigma_i^{(j)}\}_{j=1}^{m_i}$ *are the singular values of* $P_i^HQ_i$ *in which the* P_i *and* Q_i *satisfy* $\Re(P_i)=\mathscr{Y}_i$, $\Re(Q_i)=\mathscr{X}_i$ *and*

$$P_i^HP_i=Q_i^HQ_i=I^{(m_i)}, \quad i=1, 2, \cdots, r.$$

The Schur decomposition of any diagonalizable matrix has an important characteristic clarified by the following lemma.

Lemma 4.2. *Let A be an* $m\times m$ *diagonalizable matrix with Schur decomposition*

$$U^HAU=\Lambda+M\equiv T, \tag{4.10}$$

where U is a unitary matrix, M is a strictly upper triangular matrix (i.e., M is an upper triangular matrix with zeros on its diagonal) and