A SEMILINEAR FINITE ELEMENT METHOD*

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Abstract

In the Ritz-Galerkin method the linear subspace of the trial solution is extended to a closed subset. Some results, such as orthogonalization and minimum property of the error function, are obtained. A second order scheme is developed for solving a linear singular perturbation elliptic problem and error estimates are given for a uniform mesh size. Numerical results for linear and semilinear singular perturbation problems are included.

§ 1. Introduction

The development of finite element methods has been successful in various fields. From a mathematical point of view, the methods are an extension of the Rayleigh-Ritz-Galerkin technique ([1], [11]—[13]). Usual finite element schemes, choosing piecewise polynomials as trial functions, are very efficient when there are no steep gradients in the true solution. Otherwise, poor results might occur. In order to get accurate numerical data, one may use the adaptive mesh technique or a higher precision scheme such as h-version or p-version[8]. Besides usual polynomials, rational elements[17] and exponential elements[6] have been introduced to enrich the trial subspace to reduce a number of parameters for a given precision. One thing in common among these techniques is that they are all reduced to a discrete linear system if the original differential equation is linear.

This paper proposes finite element methods of Ritz and Galerkin types for linear elliptic equations where the shape functions depend nonlinearly on a finite set of parameters. So the arising minimization problem is solved on a subset instead of a linear subspace (as it would be the case for piecewise linear shape functions). This approach allows for instance the use of exponential shape functions with the parameters occurring in the exponent. So in this case one probably obtains a significantly better approximation which justifies the additional labour.

In Sections 2 and 3, we generalize respectively the Ritz and the Galerkin methods from linear trial subspaces to subsets, and derive some results such as orthogonalization and error estimates. In Section 4, the semilinear finite element technique is applied to solve singular perturbation problems in one dimension: $-\varepsilon u''+pu'+qu=f$, $u(0)=u(1)=0$, which has been studied by various authors[4-9]. Our analysis shows an improvement over the scheme of using the piecewise linear subspace by one higher order of precision. Moreover, the constraint of mesh size h is relaxed from $O(\varepsilon^2)$ to $O(\varepsilon)$. The numerical tests including a linear and a

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semilinear test singular perturbation problems are given in Section 5. Computational results show good agreement with the above theoretical analysis.

Some research results on the same topic in two-dimensions will be reported separately[16].

§ 2. A Ritz Method on Subsets

First we consider a self-adjoint elliptic linear differential equation

$$Lu=f. \tag{1}$$

Suppose $a(u, v)=(Lu, v)$ is a positive quadratic form in a real Hilbert space H with an inner product $(*, *)$ and a norm $\|*\|$:

$$C_2\|u\|^2\leqslant a(u, u)\leqslant C_1\|u\|^2 \quad \text{for all } u\in H, \tag{2}$$

where C_1 and C_2 are positive constants. u is defined as a weak solution of (1) if it satisfies

$$a(u, v)=(f, v) \quad \text{for all } v\in H. \tag{3}$$

It is well-known that u is a weak solution of (1) if and only if it is the unique minimum solution of a quadratic functional I, i.e.,

$$I(u)=\inf_{v\in H} I(v)=\inf_{v\in H}\{a(v, v)-2(f, v)\}. \tag{4}$$

In dealing with the variational problem (4), a well-known discretization is used to replace the space H with a sequence of finite-dimensional subspaces V^h contained in H such that

$$I(u^h)=\inf_{v\in V^h} I(v),$$

which is equivalent to the following weak solution

$$a(u^h, v^h)=(f, v^h) \quad \text{for all } v^h\in V^h. \tag{5}$$

Now we try to replace H in (4) with a sequence of closed subsets S^h with the same number of finite-dimensional parameters. Let T be a one-to-one differentiable map[10] from an open convex set V_1^h of V^h onto S^h: $TV_1^h=S^h$. In particular, $T'=T$ if T is a linear map.

Consider a restricted variational problem on the closed subset S^h:

$$I(u_s)=\inf_{v\in S^h} I(v). \tag{6}$$

Since S^h is closed, there exists a solution of (6) in S^h. If u_s minimizes I over S^h, $u_s=Tw$, then for any $\eta\in V_1^h$ and small $\alpha, I(u_s)\leqslant I(T(w+\alpha\eta))$ as $w+\alpha\eta\in V_1^h$. Let $T(w+\alpha\eta)=Tw+\alpha T\eta+\varkappa(\alpha)$, where T is positively homogeneous and

$$\varkappa(\alpha)=T(w+\alpha\eta)-Tw-\alpha T\eta.$$

We see that

$$I(T(w+\alpha\eta))=I(u_s)+2\alpha[a(u_s, T\eta)-(f, T\eta)]+2[a(u_s, \varkappa(\alpha))-(f, \varkappa(\alpha))]$$
$$+\alpha^2a(T\eta, T\eta)+2\alpha a(T\eta, \varkappa(\alpha))+a(\varkappa(\alpha), \varkappa(\alpha))\equiv I(\alpha).$$

For u_s to minimize I over S^h, it requires that $\lim_{\alpha\to 0} I'(\alpha)=0$. Observing that $\varkappa(0)=0$, $\varkappa'(0)=(T'(T^{-1}u_s)-T)\eta$, and