LINEAR FINITE ELEMENTS WITH HIGH ACCURACY*

LIN QUN (林 群)
(*Institute of Systems Science, Academia Sinica, Beijing, China*)

XU JIN-CHAO (许进超)
(*Peking University, Beijing, China*)

§ 1. Introduction

This paper is concerned with the superconvergence and the acceleration of finite element methods. We start with the simplest finite element method, namely the linear elements. By using the piecewise strongly regular triangulation (see Definition 3) we find that the stress in the given domain can be approximated with the accuracy $O\left(h^2\log\frac{1}{h}\right)$ (see Theorem 3.2). Furthermore, higher accuracy, like $O\left(h^3\log^2\frac{1}{h}\right)$ or $O\left(h^4\log^2\frac{1}{h}\right)$, can be achieved if the extrapolation method is adopted. It seems that the linear elements are good enough for achieving higher accuracy in some cases.

As a by-product, some posteriori error estimates for finite elements are obtained in the two-dimensional case.

The paper is built upon the previous works by Lin, Lu, Xu, Zhu (see [11—15, 22—26]). A number of important related works which have influenced our analysis are included in the bibliography.

We clarify the analysis and generalize the ideas in [11—15, 22—24]. New results as well as shorter and more revealing proofs of known results are obtained. For the sake of expository continuity, the paper is essentially self-contained.

§ 2. Some Superconvergence Estimates

For simplicity, Let us consider the model problem: Find $u\in H_0^1(\Omega)$ such that

$$-\Delta u=f \quad \text{in } \Omega, \tag{2.1}$$

where $\Omega\subset R^2$ is a convex domain with Lipschitze continuous boundary.

We will approximate (2.1) by the simplest finite element method, namely linear elements. For this, let $T_h=\{K\}$, $0<h<h_0<1$, be a finite triangulation, which is supposed to be quasi-uniform, i. e. it satisfies the following condition: Each triangle $K\in T_h$ contains a circle of radius c_1h and is contained in a circle of radius c_2h, where the constants c_1, c_2 do not depend on K or h.

For clarity, let us introduce the definitions of some special kinds of quasi-uniform triangulation:

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Definition 1. *A triangulation $T_h=\{K\}$ is called strongly regular if any two adjacent triangles of T_h form an approximate parallelogram, i.e. there exists a constant c independent of h, K, such that (see Fig. 1)*

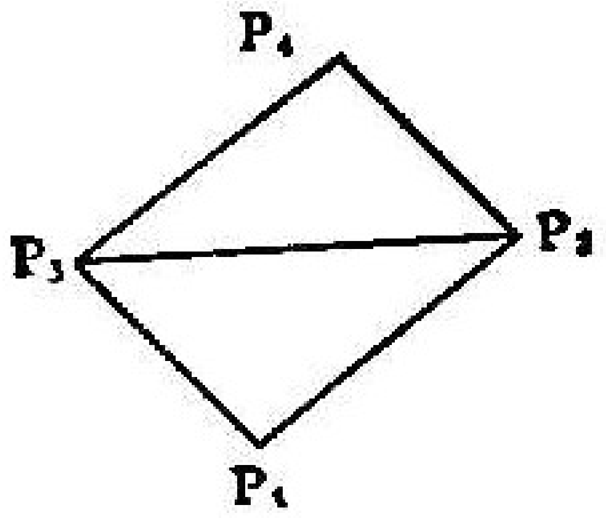

Fig. 1

$$|\overrightarrow{P_1P_2}-\overrightarrow{P_3P_4}|\leqslant ch^2. \tag{2.2}$$

Definition 2. *A triangulation is called completely regular if any two adjacent elements form an exact parallelogram.*

Generally speaking, the strong regularity condition is hard to be thoroughly satisfied over all the given region. For instance, a generic polygonal domain seemingly cannot be triangulated in the sense of strong regularity. But it is easy to observe that the strongly regular triangulation can be achieved over any quadrilateral or triangular region. For this reason we introduce the following

Definition 3. *A triangulation $T_h=\{K\}$ on the polygonal domain is called piecewise strongly regular, if Ω is divided into some quadrilateral or triangular subdomains with the vertices at the boundary, and the triangulation restricted on each such subdomain is strongly regular.*

Let S^h be a piecewise linear finite element space on Ω_h with zero on $\Omega\backslash\Omega_h$, and $u^h\in S^h$ the finite element approximation satisfying

$$a(u^h,\ v^h)=(f,\ v^h),\quad \forall v^h\in S^h. \tag{2.3}$$

For any fixed $z_0\in\Omega$, the Green function $G_{z_0}\in H_0^{1,1}(\Omega)$ is defined by

$$a(G_{z_0},\ v)=v(z_0),\quad \forall v\in C_0^\infty(\Omega) \tag{2.4}$$

and its finite element approximation $G_{z_0}^h\in S^h$ by

$$a(G_{z_0}^h,\ v^h)=v^h(z_0),\quad \forall v^h\in S^h. \tag{2.5}$$

The following estimates (for quasi-uniform triangulation) are shown by Frehse, Rannacher and Scott[6,16], Schatz and Wahlbin[17], Zhu[24]:

$$\|G_{z_0}-G_{z_0}^h\|_{1,1,\Omega}\leqslant ch\log\frac{1}{h}, \tag{2.6}$$

$$\|G_{z_0}-G_{z_0}^h\|_{0,1,\Omega}\leqslant ch^2\log^2\frac{1}{h}, \tag{2.7}$$

$$|(G_{z_0}-G_{z_0}^h)(z_1)|\leqslant ch^2\left|\log\frac{|z_1-z_0|}{h}\right|\Big/|z_1-z_0|^2, \tag{2.8}$$

$$|G_{z_1}^h-G_{z_2}^h|_{1,1,\Omega}\leqslant ch\log\frac{1}{h}, \tag{2.9}$$

$$\|G_{z_0}^h\|_{0,\infty,\Omega}+\left(\log\frac{1}{h}\right)^{\frac{1}{2}}\|G_{z_0}^h\|_{1,2,\Omega}\leqslant c\log\frac{1}{h}, \tag{2.10}$$

where $z_1,\ z_2\in\Omega$ with $|z_1-z_2|=O(h)$.

It is also known that there exists $q_0=q_0(\Omega)\in(2,\ \infty)$ such that if $q\in[2,\ q_0)$, then for all $F\in L^q(\Omega)$, there exists $v\in H^{2,q}(\Omega)\cap H_0^1(\Omega)$ such that

$$-\Delta v=F\quad \text{in } \Omega,$$

$$\|v\|_{2,q,\Omega}\leqslant C\|F\|_{0,q,\Omega}.$$

Lemma 2.1. *For each $p\in[1,\ 2)$ there exists a constant $c(p)>0$ such that*