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ON ERROR ESTIMATE OF SPACE-TIME FINITE ELEMENT METHODS FOR PARABOLIC EQUATIONS IN A TIME-DEPENDENT DOMAIN*

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Abstract

We consider linear parabolic equations in a space-time domain with curved boundaries and nonhomogeneous Dirichlet boundary conditions and discuss their approximations with isoparametric space-time finite elements. A general error estimate is proved and applied to some elements of practical interest.

1. Introduction

The front-tracking methods using space-time finite elements are very effective in solving moving boundary problems, as shown by numerical experiments [Bonnerot and Jamet (1974, 1977, 1979, 1981); Li (1982, 1983)]. During the solution process, the original problem is reduced to two coupled problems: determination of the position of the moving boundary and solution of the parabolic equation in a known space-time domain with curved boundaries.

As a first step towards the complete error analysis of the front-tracking method, we want to obtain the error estimate of the approximation of the parabolic equation in a known curved space-time domain. Jamet (1978) considered the case with polygonal boundaries and homogeneous Dirichlet boundary conditions. In solving the moving boundary problems, however, it is not an appropriate approach to transform a problem with nonhomogeneous boundary data into a problem with homogeneous boundary data before discretization, since the position of the moving boundary is not known a priori, and such a transformation will greatly complicate the problem. Moreover, in most cases, the moving boundary is not polygonal. It is, therefore, necessary to consider the direct discretization of the parabolic equation in a curved space-time domain with nonhomogeneous boundary conditions. Such a discretization method will be described in section 2 of this paper. A general error estimate of the approximation will be proved in section 3. It is then applied to some finite elements of practical interest in section 4. We will follow Jamet's technique in the proof of the general error estimate. The results of this paper are extensions of his results.

2. Description of the Discretization Method

Consider a time interval $[0, T]$. Let $\Omega(t)$ be a bounded domain in R^m and

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$\Gamma(t)$ be its boundary. Let $R^T=\{(P, t);\ P\in\Omega(t), 0<t\leqslant T\}$ be a space-time domain. $\Sigma^T=\{(P, t);\ P\in\Gamma(t),\ 0\leqslant t\leqslant T\}$ is its lateral boundary.

We consider the following problem:

$$\frac{\partial u}{\partial t}-\nabla^2 u=f \quad \text{in } R^T, \tag{2.1}$$

$$u=g \quad \text{in } \Sigma^T, \tag{2.2}$$

$$u=u^0 \quad \text{in } \Omega(0), \tag{2.3}$$

where $f\in L^2(R^T)$, $u^0\in L^2(\Omega(0))$ and g is a continuous function on Σ^T. ∇u is the gradient of u with respect to the space variables.

Let $G=G(t_1, t_2)=\{(P, t);\ P\in\Omega(t),\ 0\leqslant t_1<t<t_2\leqslant T\}$ be a subdomain of R^T and $\Phi(G)$ be the space of all Lipschitz-continuous functions defined on $\bar G$ which vanish on the lateral boundary of G. Then a classical solution u of (2.1)—(2.3) is also the weak solution defined by

$$B_G(u, \phi)=-\left(\left(u, \frac{\partial\phi}{\partial t}\right)\right)_G+((\nabla u, \nabla\phi))_G+(u, \phi)_{\Omega(t_2)}-(u, \phi)_{\Omega(t_1)}=((f, \phi))_G \tag{2.4}$$

for all $\phi\in\Phi(G)$ and for all $0\leqslant t_1<t_2\leqslant T$, together with the initial-boundary conditions (2.2)—(2.3).

Here we have used the notations

$\Omega(t)=$ section $\{(P, t);\ p\in\Omega(t)\}$, $(\cdot,\cdot)_{\Omega(t)}=$ inner product in $L^2(\Omega(t))$,

$((\cdot, \cdot))_G=$ inner product in $L^2(G)$, $((\nabla u, \nabla\phi))_G=\iint_G \nabla u\cdot\nabla\phi\, d\underset{\sim}{x}\, dt.$

To define the approximate solution, we consider a subdivision of $[0, T]$: $0=t^0<t^1<\cdots<t^n<\cdots<t^N=T$. Let Σ_h^T be a continuous and piecewise smooth approximation to Σ^T. Let G_h^n be the space-time domain bounded by Σ_h^T and the hyperplanes $t=t^n$ and $t=t^{n+1}$. Let $R_h^T=\bigcup_{n=0}^{N-1}\bar G_h^n$. We assume that there exists a bounded domain $\tilde R^T$ such that $\tilde R^T\supset R^T$ and $\tilde R^T\supset R_h^T$ for all small enough values of h, the discretization parameter, and for all subdivisions of $[0, T]$. Assume also that the functions f, u^0 and the exact solution u have smooth enough extensions $\tilde f$, $\tilde u^0$ and $\tilde u$ to $\tilde R^T$ which also satisfy (2.1) and (2.3).

Let $\Omega_h^n=\Omega_h(t^n)$ be the section of R_h^T on the hyperplane $t=t^n$ and $\underset{\sim}{G}_h^n=\bar G_h^n-\bar\Omega_h^n$. Let Φ_h^n be a finite dimensional subspace of $\Phi(G_h^n)$, $1\leqslant n\leqslant N-1$, and V_h be the space of all functions v_h defined on R_h^T such that their restriction to each $\underset{\sim}{G}_h^n$ coincides with the restriction of a function $q^{(n)}v_h\in\Phi_h^n$ to $\underset{\sim}{G}_h^n$. Let also $U_h=w_h+V_h$ where w_h is a given function defined to R_h^T, which is Lipschitz-continuous on each $\underset{\sim}{G}_h^n$ and whose restriction to Σ_h^T is an appropriate approximation of the restriction of $\tilde u$ to Σ_h^T. Note that the functions $v_h\in V_h$ and $u_h\in U_h$ are in general discontinuous at time $t=t_n$, $0\leqslant n\leqslant N-1$. We denote $v_h(\cdot, t^n)$ and $v_h(\cdot, t^{n+0})$ by v_h^n and v_h^{n+0} respectively.

Now we can define the discrete problem as follows:

Find $u_h\in U_h$ such that $u_h^0=\tilde u^0|_{\Omega_h^0}$ and

$$B_{G_h^n}(u_h, \phi_h)=((\tilde f, \phi_h))_{G_h^n}, \tag{2.5}$$

for all $\phi_h\in\Phi_h^n$ and for all $0\leqslant n\leqslant N-1$.