A NUMERICAL ALGORITHM FOR AN INVERSE PROBLEM OF A PARTIAL DIFFERENTIAL EQUATION WITH MULTI-PARAMETER TO BE DETERMINED*

LIU JIA-QI (刘家琦)

(Department of Applied Mathematics, Harbin Institute of Technology, Harbin, China)

Abstract

A formulation of an inverse problem of a partial differential equation with multi-parameter to be determined is introduced. The numerical algorithm, pulse-spectrum technique, is extended to solve this type of inverse problem. An example for remote sensing of the thermal conductivity and specific heat of a nonhomogeneous material is demonstrated. Numerical simulations are carried out to test the feasibility and to study the general characteristics of this technique without real measurement data. It is found that the extended pulse-spectrum technique gives excellent results.

1. Introduction

In this paper, we consider an inverse problem of determining the coefficients of a partial differential operator with a known structure in terms of a governing equation, some initial conditions and boundary conditions, and some auxiliary information. We call this type of inverse problem "operator identification". It originates from various kinds of problems of mathematical physics, for example, heat transfer, structural dynamics, engineering synthesis, remote sensing, geophysical prospecting, medical diagnostics and so on.

In section 2, the inverse problem of a partial differential equation with multi-parameter to be determined is formulated and the pulse-spectrum technique (PST) is extended to solve this type of inverse problem. The PST was first introduced by Tsien and Chen[1] to solve an idealized one-dimensional velocity inversion problem in fluid dynamics. It was further developed to handle the noisy, poorly distributed and inadequately measured data by Chen and Tsien[10]. Later it was used to solve a one-dimensional inverse problem in electro-magnetic wave propagation by Tsien and Chen[11]. In a different direction, the PST was modified to solve an inverse problem of a one-dimensional diffusion equation[12]. The basic idea of the PST is to measure data in the time domain as functions which are Laplace transformable, and carry out numerically the synthesis of the unknown parameter in the complex frequency domain by a special iterative algorithm.

As an example, the remote sensing of the thermal conductivity and specific heat of a nonhomogeneous material is demonstrated. Numerical simulations are carried out to test the feasibility and to study the general characteristics of this

* Received September 7, 1982.

technique without real measurement data. It is found that the extended PST does give excellent results for the inverse problem of multi-parameter to be determined.

2. Formulation of the Inverse Problem and Numerical Algorithm

Consider the following differential equation defined on the time-space domain

$$Lu(x, t)=\varphi(x, t), \quad (x, t)\in\Omega\times T, \tag{1}$$

where $x=(x_1, x_2, \cdots, x_p)$, Ω is a p-dimensional domain, $\partial\Omega$ is the boundary of Ω, $T=[t|0<t]$, $u(x, t)$ is a sufficiently smooth function defined on $\Omega\times T$, which is Laplace transformable with respect to time variable t. Furthermore we suppose that the differential operator has the following form (structure)

$$L=\sum_{k=1}^{m}[A_k(\alpha_k(x)C_k)+\beta_k(x)D_k], \tag{2}$$

where A_k, C_k and D_k are elementary linear operators (for example, differentiation, integration, or their various combinations). A_k and C_k are operators of variable x and D_k is an operator of variable t. The coefficient $\alpha_k(x)$ is a piecewise smooth function on Ω; $\beta_k(x)$ is a piecewise continuous function on Ω.

The inverse problem of the differential Eq. (1), i. e. operator identification, can be described as follows: The governing equation will be in the form

$$Lu(x, t)=\varphi(x, t), \quad (x, t)\in\Omega\times T \tag{3}$$

with initial condition as

$$Eu(x, 0)=0, \quad x\in\Omega \tag{4}$$

and boundary condition as

$$Bu(x, t)=f(x, t), \quad x\in\partial\Omega,\ 0<t \tag{5}$$

and auxiliary boundary conditions as

$$B_iu(x, t)=f_i(x, t), \quad x\in\partial\Omega_i,\ i=1, 2, \cdots, I, \tag{6}$$

where E, B and B_i are initial operator, boundary operator and auxiliary boundary operator respectively. In Eq. (6) I is the number of auxiliary boundary conditions, and $\partial\Omega_i$ is a part of boundary $\partial\Omega$.

The inverse problem is to determine the coefficients $\alpha_k(x)$, $\beta_k(x)$ $(k=1, 2, \cdots, m)$ of the unknown operator from known operators A_k, C_k, D_k, B, B_i, E and Eqs. (3)—(6).

Numerical Algorithm. The PST calls for the Laplace transformation of (3), (5), (6). So the problem is transformed from the time domain to the complex frequency domain and the corresponding problem is

$$\sum_{k=1}^{m}[A_k(\alpha_k(x)c_k)+\beta_k(x)P(s)]U(x, s)=\Phi(x, s), \tag{7}$$

$$\bar{B}U(x, s)=F(x, s), \quad x\in\partial\Omega, \tag{8}$$

$$\bar{B}_iU(x, s)=F_i(x, s), \quad x\in\partial\Omega_i,\ i=1, 2, \cdots I, \tag{9}$$

where $P(s)$ is the polynomial of frequency s, and $U(x, s)$, $\Phi(x, s)$, $F(x, s)$, $F_i(x, s)$, $\bar{B}U(x, s)$ and $\bar{B}_iU(x, s)$ are Laplace transformations of $u(x, t)$, $\varphi(x, t)$, $f(x, t)$, $f_i(x, t)$, $Bu(x, t)$ and $B_iu(x, t)$ respectively.

Now, the synthesis is carried out in the (x, s) domain, and the unknown