A FINITE ELEMENT APPROXIMATION OF NAVIER-STOKES EQUATIONS USING NONCONFORMING ELEMENTS*

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§1. Introduction

In this paper, steady incompressible flow of viscous fluids is considered. The finite element approximation of this problem has been treated by some authores[1-3]. By means of a primitive variable formulation, the numerical treatment of Navier-Stokes equations naturally leads to the mixed finite element method, in which the Babuska-Brezzi condition is required for the conforming finite element method. It means that the finite dimensional subspace of the velocity field and the subspace of pressure must satisfy a certain matchable relationship. For example, in the two dimensional case, triangular elements are used; the subspace of the velocity field is formed by piecewise linear functions and the subspace of pressure is formed by piecewise constant functions for a conforming finite element method. It is straightforward to show that they do not satisfy Babuska-Brezzi condition. If the subspace of the velocity field is formed by piecewise quadratic functions instead of piecewise linear functions, then they satisfy the Babuska-Brezzi condition. However, a loss of precision is also incurred. The optimal error estimate cannot be obtained in this situation. M. Crouzeix and P. A. Raviart proposed to use the nonconforming triangular elements to form the approximation space of velocity field to solve stokes equations. A few years later, the nonconforming triangular elements were applied to stationary Navier-Stokes equations by R. Temam[1] and to nonstationary Navier-Stokes equations by R. Rannacher[5]. In those cases the optimal error estimate can be obtained, and therefore using nonconforming elements may be a good choice for the finite element approximation of Navier-Stokes equations. Recently a class of nonconforming rectangular elements were used for the numerical analysis of stokes equations and an optimal error estimate was given[6]. The aim of this paper is to analyse the error estimate of a finite element approximation of Navier-Stokes equations using general nonconforming elements including nonconforming rectangular elements.

Let Ω be a bounded domain of $R^n (n=2, 3)$ with a Lipschitz continuous boundary $\partial\Omega$. Let $W^{m,p}(\Omega)$ denote the Sobolev space on Ω with norm $\|\cdot\|_{m,p,\Omega}$. As usual, when $p=2$, $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$; when $m=0$, $W^{0,p}(\Omega)$ is denoted by $L_p(\Omega)$. Moreover, let $H_0^1(\Omega)=\{u\in H^1(\Omega),\ u=0 \text{ on } \partial\Omega\}$, $X=(H_0^1(\Omega))^n$ with norm $\|\cdot\|_X=|\cdot|_{1,2,\Omega}$ and $M=\left\{\lambda\in L_2(\Omega), \int_\Omega \lambda\, dx=0\right\}$ with norm $\|\cdot\|_M=\|\cdot\|_{0,2,\Omega}$.

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We consider the following boundary value problem of Navier-Stokes equations:

$$-\nu\Delta \boldsymbol{u}+\sum_{j=1}^{n} u_j \frac{\partial \boldsymbol{u}}{\partial x_j}+\operatorname{grad} \lambda=\boldsymbol{f}, \quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u}=0, \quad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{u}=0, \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\boldsymbol{u}=(u_1, \cdots, u_n)$ is the velocity vector, λ is the pressure, and ν is a positive constant, the coefficient of kinematic viscosity. Eq. (1.1) can be rewritten as

$$-\nu\Delta \boldsymbol{u}+\frac{1}{2}\sum_{j=1}^{n} u_j \frac{\partial \boldsymbol{u}}{\partial x_j}+\frac{1}{2}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(u_j\boldsymbol{u})+\operatorname{grad} \lambda=\boldsymbol{f}, \quad \text{in } \Omega. \tag{1.1$'$}$$

Then the boundary value problem (1.1)′, (1.2), (1.3) is equivalent to the following variational problem:

Find $(\boldsymbol{u}, \lambda)\in X\times M$, such that

$$a_0(\boldsymbol{u}, \boldsymbol{v})+a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v})+b(\boldsymbol{v}, \lambda)=\langle \boldsymbol{f}, \boldsymbol{v}\rangle, \quad \forall \boldsymbol{v}\in X, \tag{1.4}$$

$$b(\boldsymbol{u}, \mu)=0, \quad \forall \mu\in M, \tag{1.5}$$

where

$$a_0(\boldsymbol{u}, \boldsymbol{v})=\nu\sum_{i,j=1}^{n}\int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}dx, \tag{1.6}$$

$$a_1(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})=\frac{1}{2}\sum_{i,j=1}^{n}\int_\Omega w_j\left(\frac{\partial u_i}{\partial x_j}v_i-\frac{\partial v_i}{\partial x_j}u_i\right)dx, \tag{1.7}$$

$$b(\boldsymbol{v}, \lambda)=-\int_\Omega \lambda \operatorname{div} \boldsymbol{v}\, dx, \tag{1.8}$$

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle=\sum_{i=1}^{n}\int_\Omega f_i v_i\, dx. \tag{1.9}$$

Obviously,

$$a_1(\boldsymbol{w}; \boldsymbol{v}, \boldsymbol{v})=0, \quad \forall \boldsymbol{w}, \boldsymbol{v}\in X. \tag{1.10}$$

Let $V=\{\boldsymbol{v}\in X, \operatorname{div} \boldsymbol{v}=0\}$, and

$$N=\sup_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V}\frac{|a_1(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})|}{\|\boldsymbol{w}\|_X\|\boldsymbol{u}\|_X\|\boldsymbol{v}\|_X}, \tag{1.11}$$

$$\|\boldsymbol{f}\|^*=\sup_{\boldsymbol{v}\in V}\frac{|\langle \boldsymbol{f}, \boldsymbol{v}\rangle|}{\|\boldsymbol{v}\|_X}. \tag{1.12}$$

Suppose $\boldsymbol{f}\in(H^{-1}(\Omega))^n$, and $\frac{N\|\boldsymbol{f}\|^*}{\nu^2}<1$. Then problem (1.4)—(1.5) has a unique solution $(\boldsymbol{u}, \lambda)\in X\times M$ (see [3]). In this paper, we restrict ourselves within this case.

§ 2. An Abstract Error Estimate

Let $H=(L_2(\Omega))^n$. Throughout this section we suppose $\boldsymbol{f}\in H$. For each $h>0$, let M^h and X^h be two finite dimensional spaces such that $M^h\subset M$ and $X^h\subset H$, but X^h is not a subspace of X in the general case. Let $\|\cdot\|_h$ denote the norm of X^h (examples of X^h will be given in the next section). In order to discretize the variational problem (1.4)—(1.5), we first extend the definitions of $a_0(\boldsymbol{u}, \boldsymbol{v})$, $a_1(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})$ and $b(\boldsymbol{v}, \lambda)$ to $(X\cup X^h)^2$, $(X\cup X^h)^3$ and $(X\cup X^h)\times M$ and denote them as $a_0^h(\boldsymbol{u}, \boldsymbol{v})$, $a_1^h(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})$ and $b^h(\boldsymbol{v}, \lambda)$ and

$$a_0^h(\boldsymbol{u}, \boldsymbol{v})=a_0(\boldsymbol{u}, \boldsymbol{v}), \quad \forall \boldsymbol{u}, \boldsymbol{v}\in X,$$

$$a_1^h(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})=a_1(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}), \quad \forall \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}\in X,$$