ASYMPTOTIC RADIATION CONDITIONS FOR REDUCED WAVE EQUATION*

FENG KANG (冯 康)
(*Computing Center, Academia Sinica, Beijing, China*)

Abstract

In this note the exact non-local radiation condition and its local approximations at finite artificial boundary for the exterior boundary value problem of the reduced wave equation in 2 and 3 dimensions are discussed. Based on the asymptotic expansion of Hankel functions for large arguments, an approach for the construction of local approximations is suggested and gives expression of the normal derivative at spherical artificial boundary in terms of linear combination of Laplace-Beltrami operator and its iterates, i.e. tangential derivatives of even order exclusively. The resulting formalism is compatible with the usual variational principle and the finite element methodology and thus seems to be convenient in practical implementation.

Boundary value problems of P. D. E. involving infinite domain occur in many areas of applications, e. g., fluid flow around obstacles, coupling of structures with foundation and environment, scattering and radiation of waves and so on. For the numerical solution of this class of problems, the natural approach is to cut off an infinite part of the domain and to set up, at the computational boundary of the remaining finite domain, appropriate artificial boundary conditions. In the usual treatment, the latter is carried out, however, in an oversimplified way without sufficient justification. Along this line of approach, there is recent interest and progress leading to a latter understanding of the nature of the problem and several sequences of improved artificial boundary conditions. In the following we shall discuss briefly, for the reduced wave equation with spherical computational boundary, the exact integral boundary condition at the spherical computational boundary, and suggest, using asymptotic expansions of Hankel functions, a method for deriving a sequence of approximations of the non-local boundary operator by means of tangential differential operators on the boundary, in a form which is compatible with the variational form and the finite element method for the original problem.

I

The general solution of the 2-D reduced wave (Helmholtz) equation

$$\Delta_2 u+\omega^2 u=0, \quad \Delta_2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \tag{1.1}$$

in the exterior $\Omega_a=\{r>a\}$ to the circle $\Gamma_a=\{r=a\}$ of radius a satisfying the radiation condition at infinity

* Received March 19, 1983.

$$u=O(r^{-1}),$$
$$u_r+i\omega u=o(r^{-\frac{1}{2}}) \quad \text{as} \quad r\to\infty \tag{1.2}$$

can be represented as a Fourier series

$$u(r,\ \theta)=\sum_{-\infty}^{\infty} A_n H_n^{(2)}(\omega r)e^{in\theta}, \tag{1.3}$$

where $H_n^{(2)}$ is the Hankel function of the 2nd kind of order n, and in particular,

$$u(a,\ \theta)=\sum_{-\infty}^{\infty} A_n H_n^{(2)}(\omega a)e^{in\theta}. \tag{1.4}$$

So (1.3) can be written as

$$u(r,\ \theta)=\sum_{-\infty}^{\infty}\left(\frac{H_n^{(2)}(\omega r)}{H_n^{(2)}(\omega a)}\right)A_n H_n^{(2)}(\omega a)e^{in\theta}.$$

(1.4) and (1.5) together give the Poisson integral formula

$$u=P\hat{u} \tag{1.5}$$

expressing the solution $u=u(r,\ \theta)$ in domain Ω_a in terms of its Dirichlet data $\hat{u}=u(a,\ \theta)$ on boundary Γ_a, or explicitly

$$u(r,\ \theta)=P(\theta)*u(a,\ \theta)=\int_0^{2\pi}P(\theta-\theta')u(a,\ \theta')d\theta',\quad 0\leqslant\theta\leqslant 2\pi,\ r>a, \tag{1.6}$$

where

$$P(\theta)=\frac{1}{2\pi}\sum_{-\infty}^{\infty}\left(\frac{H_n^{(2)}(\omega r)}{H_n^{(2)}(\omega a)}\right)e^{in\theta}, \tag{1.7}$$

$*$ denotes the circular convolution

$$f(\theta)*g(\theta)=\int_0^{2\pi}f(\theta-\theta')g(\theta')d\theta'.$$

Differentiation of (1.3) gives

$$u_r(r,\ \theta)=\sum_{-\infty}^{\infty}A_n\omega H_n^{(2)\prime}(\omega r)e^{in\theta} \tag{1.8}$$

and

$$u_r(a,\ \theta)=\sum_{-\infty}^{\infty}A_n\omega H_n^{(2)\prime}(\omega a)e^{in\theta}=\sum_{-\infty}^{\infty}\left(\omega\frac{H_n^{(2)\prime}(\omega a)}{H_n^{(2)}(\omega a)}\right)A_nH_n^{(2)}(\omega a)e^{in\theta}. \tag{1.9}$$

(1.4) and (1.9) together give the integral relation

$$\hat{u}_\nu=-\hat{u}_r=\boldsymbol{K}\hat{u} \tag{1.10}$$

expressing the solution's normal derivative $\hat{u}_\nu=-\hat{u}_r=u_r(a,\ \theta)$ (ν is directed to the exterior of the domain Ω_a), i.e. the Neumann data on Γ_a, of the solution u in terms of the corresponding Dirichlet data $\hat{u}$, or explicitly

$$-u_r(a,\ \theta)=K(\theta)*u(a,\ \theta)=\int_0^{2\pi}K(\theta-\theta')u(a,\ \theta')d\theta', \tag{1.11}$$

where

$$K(\theta)=\frac{1}{2\pi}\sum_{-\infty}^{\infty}\left(-\omega\frac{H_n^{(2)\prime}(\omega a)}{H_n^{(2)}(\omega a)}\right)e^{in\theta}. \tag{1.12}$$

The integral in (1.11) is highly singular of non-integrable type, it is to be understood in the sense of regularization of divergent integrals in the theory of distributions. $\boldsymbol{K}$ is in fact a pseudo-differential operator of order 1 on the boundary maniford Γ_a and defines a linear continuous map