NONCONFORMING ELEMENTS IN THE MIXED FINITE ELEMENT METHOD*

HAN HOU-DE (韩厚德)

(*Department of Mathematics, Peking University, Beijing, China*)

Abstract

In this paper, an abstract error estimate of mixed finite element methods using nonconforming elements is presented. In addition, a class of nonconforming rectangular elements is proposed, and applied to Stokes equations. The optimal error estimate is given.

I. Introduction

The mixed finite element method has been applied to many difference fields, such as solid mechanics, fluid mechanics, and so on. The study of this method can be reduced to the following abstract saddle-point problem. Let V and W be two real Hilbert spaces, whose norms, scalar products and dual spaces are denoted by $\|\cdot\|_V$, $(\cdot,\ \cdot)_V$, V' and $\|\cdot\|_W$, $(\cdot,\ \cdot)_W$, W' respectively. Let $\langle\cdot,\ \cdot\rangle$ denote duality between both V' and V and W' and W. The variational problem is

Find $(u,\ p)\in V\times W$, such that

$$\begin{cases} a(u,\ v)+b(v,\ p)=\langle f,\ v\rangle, & \forall v\in V, \\ b(u,\ q)=\langle g,\ q\rangle, & \forall q\in W, \end{cases} \tag{1.1}$$

where $a(\cdot,\ \cdot)$ and $b(\cdot,\ \cdot)$ are bounded bilinear forms on $V\times V$ and $V\times W$, respectively, and $f\in V'$, $g\in W'$ are given.

Given two finite dimensional subspaces $V_h\subset V$ and $W_h\subset W$, $0<h\leqslant h_0$, the finite element approximation of (1.1) is the solution of the following problem:

Find $(u_h,\ p_h)\in V_h\times W_h$, such that,

$$\begin{cases} a(u_h,\ v)+b(v,\ p_h)=\langle f,\ v\rangle, & \forall v\in V_h, \\ b(u_h,\ q)=\langle g,\ q\rangle, & \forall q\in W_h. \end{cases} \tag{1.2}$$

In 1974, Brezzi[1] studied the saddle-point problem (1.1) and its finite element approximation. The main results are the following:

Let $Z=\{v\in V;\ b(v,\ q)=0,\ \forall q\in W\}$. If

(i) there is a constant $\alpha>0$, such that

$$a(v,\ v)\geqslant\alpha\|v\|_V^2,\quad \forall v\in Z, \tag{1.3}$$

(ii) there exists a constant $\beta>0$, such that

$$\operatorname*{Inf}_{q\in W\setminus\{0\}}\ \operatorname*{Sup}_{v\in V\setminus\{0\}} \frac{b(v,\ q)}{\|v\|_V\|q\|_W}\geqslant\beta, \tag{1.4}$$

then problem (1.1) has a unique solution $(u,\ p)$.

Let $Z_h=\{v\in V_h;\ b(v,\ q)=0,\ \forall q\in W_h\}$. If

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(i) Z_h is not empty,

(ii) there is a constant $\alpha^*>0$ independent of h, such that

$$a(v, v)\geqslant\alpha^*\|v\|_V^2, \quad \forall v\in Z_h, \tag{1.3$_h$}$$

(iii) there exists a constant $\beta^*>0$ independent of h, such that

$$\operatorname*{Inf}_{q\in W_h\setminus\{0\}} \operatorname*{Sup}_{v\in V_h\setminus\{0\}} \frac{b(v, q)}{\|v\|_V\|q\|_W}\geqslant\beta^*, \tag{1.4$_h$}$$

then problem (1.2) has a unique solution, and the following error estimates hold:

$$\|u-u_h\|_V+\|p-p_h\|_W\leqslant C\{\operatorname*{Inf}_{v_h\in V_h}\|u-v_h\|_V+\operatorname*{Inf}_{q_h\in W_h}\|p-q_h\|_W\}, \tag{1.5}$$

where C is a constant independent of h.

To get the optimal error estimates of the mixed finite element approximation of problem (1.1), we must choose finite dimensional subspaces V_h and W_h carefully. Firstly, V_h and W_h should approximate V and W with the same order of precision. Secondly, V_h and W_h must satisfy the compatibility condition $(1.4)_h$. Babuska[2] studied finite element approximation of general variationally posed problems. The major results of [2] apply to problems (1.1) and (1.2); two similar conditions equivalent to (1.4) and $(1.4)_h$ respectively can be obtained. In general, conditions (1.4) and $(1.4)_h$ are called Babuska–Brezzi conditions.

The boundary value problem of Stokes equations is a model of the abstract variational problem (1.1). Suppose $\Omega\in\mathbb{R}^2$ is a rectangular domain with boundary Γ. We consider the boundary value problem of Stokes equations:

$$\begin{cases} -\mu\Delta u+\operatorname{grad} p=f, & \text{on } \Omega,\\ \operatorname{div} u=0, & \text{on } \Omega,\\ u|_\Gamma=0, \end{cases} \tag{1.6}$$

where $u=(u_1, u_2)$ is the velocity vector, p is the pressure, μ is a positive constant, the coefficient of kinematic viscosity. Let $V=H_0^1(\Omega)\times H_0^1(\Omega)$, $W=H^0(\Omega)/P_0(\Omega)$, where $H_0^1(\Omega)$, $H^0(\Omega)$ denote the usual Sobolev spaces on Ω, and $P_0(\Omega)$ denotes the space of all constants on Ω. Then the boundary value problem (1.6) is equivalent to the following variational problem:

Find $(u, p)\in V\times W$, such that

$$\begin{cases} a(u, v)+b(v, p)=\langle f, v\rangle, & \forall v\in V,\\ b(u, q)=0, & \forall q\in W, \end{cases} \tag{1.7}$$

where $$a(u, v)=\mu\ (\operatorname{grad} u, \operatorname{grad} v)_\Omega\equiv\mu\sum_{i,j=1}^{2}\left(\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j}\right)_{H^0(\Omega)},$$

$$b(v, p)=-(\operatorname{div} v, p)_\Omega\equiv-(\operatorname{div} v, p)_{H^0(\Omega)}.$$

For example, Ω is divided into nine equal rectangular elements. The subspace of velocity field V_h is formed by piecewise bilinear functions and the subspace of pressure W_h is formed by piecewise constants, but up to now, it is not known whether subspaces V_h and W_h satisfy the B–B condition or not[3]. Some finite element schemes satisfying the B–B condition have been proposed by many authors. For instance, quadratic conforming triangular elements for the velocity field and piecewise constant triangular elements for the pressure were used by Crouzeix and Raviart[4]. Their corresponding subspaces V_h and W_h satisfy the B–B condition, but with a loss of precision of the finite