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CALCULATION OF NUMERICAL INTEGRATION OF MULTIPLE DIMENSIONS BY DISSECTION INTO SIMPLICES*

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Abstract

In this paper, we introduce a method of numerical integration on a polytope of multiple dimensions. Its basic idea is to cut a polytope into some simplices and sum the integral values on the simplices. We give the integral formulae, the expressions to estimate the error and the method for cutting a polytope into simplices. Some examples are provided to explain the adaptability of the method.

I. Introduction

For numerical integration in higher dimensions with certain complicated regions of integration and integrands it is too difficult to obtain a very precise result, because that generally requires a very large amount of computation. Some commonly used methods[1] are the network method of number theory and the Monte Carlo methods. The former has its advantage when the integrand belongs to the $E_s^\alpha(c)$ class of functions on a unit cube[2]. The Latter[3] are relatively good when high precision is not required. Their computational formulae and programs are not very complicated, but it is too difficult to obtain a precise result. In practice, sometimes we could not improve the computational precision even by one significant digit, although astonishingly large amount of additional computation is done.

In this paper, we try to study the numerical integration for an n-dimensional polytope, whose boundary consists of $(n-1)$-dimensional hyperplanes. For the integral on this kind of region we give a numerical method with relatively high precision. Its basic idea is to separate the polytope into simplices. We can adopt the centroid method of numerical integration[4] to compute the integrals on the simplices and then sum the integrals to obtain the integral on the polytope. Compared with the other methods, its computational precision is relatively high and its computational amount relatively small. For some cases the advantage of the integration method of dissection into simplices is without comparison.

II. Numerical Integration on a Simplex

Definition 2.1. *In R^n (n-dimensional Euclidean space) the convex span of $r+1$ ($r\leqslant n$) affinely independent points is an r-dimensional simplex, denoted by $r-S$. These $r+1$ independent points are the vertices of $r-S$*[5].

Assume the i-th vertex is denoted by

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$$X_i=\{x_{i1},\ x_{i2},\ \cdots,\ x_{in}\}.$$

Then, its $r+1$ vertices define a matrix

$$M=\begin{bmatrix} 1 & x_{01} & x_{02} & \cdots & x_{0n} \\ 1 & x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & & & & \\ 1 & x_{r1} & x_{r2} & \cdots & x_{rn} \end{bmatrix}.$$

From the properties of the affinely independent points, we have

$$\operatorname{rank} M=r+1.$$

Let
$$S_r=S_r(X_0,\ X_1,\ \cdots,\ X_r)$$
be the closed region of a $r-S$. Let $|S_r|$ denote the Euclidean measurement volume of $r-S$[6]. Then

$$|S_r|=\frac{1}{r!}\,[\det(MM^\tau)]^{\frac{1}{2}}. \tag{2.1}$$

If $r=n$,

$$|S_n|=\frac{1}{n!}\det M. \tag{2.2}$$

The centroid point of a $r-S$ is denoted by $\overline{X}$.

$$\overline{X}=\frac{1}{r+1}\sum_{i=0}^{r}X_i, \tag{2.3}$$

which satisfies

$$\int_{S_r}(X-\overline{X})dX=0.$$

Assume

$$I=\int_{S_r}f(X)dX. \tag{2.4}$$

When $f(X)$ has a continuous $(p+1)$-th derivative with respect to X, we can use the Taylor expansion; then

$$I=|S_r|f(\overline{X})+\frac{1}{2!}\int_{S_r}\left[(X-\overline{X})^\tau\frac{\partial}{\partial X}\right]^2 f(\overline{X})dX+\cdots$$
$$+\frac{1}{P!}\int_{S_r}\left[(X-\overline{X})^\tau\frac{\partial}{\partial X}\right]^p f(\overline{X})dX+\frac{1}{(p+1)!}\int_{S_r}\left[(X-\overline{X})^\tau\frac{\partial}{\partial X}\right]^{p+1}f(\xi)dX,$$

where

$$\frac{\partial\cdot}{\partial X}=\begin{bmatrix}\dfrac{\partial\cdot}{\partial x_1}\\ \dfrac{\partial\cdot}{\partial x_2}\\ \vdots\\ \dfrac{\partial\cdot}{\partial x_n}\end{bmatrix}.$$

We simplify this to

$$I=|S_r|f(\overline{X})+\sum_{i=2}^{p}T_i+R_{p+1}, \tag{2.5}$$

where
$$T_i=\sum_{\nu}^{|\nu|=i}\frac{m(\nu)}{\nu!}\left.\frac{\partial^i f(X)}{\partial x_1^{\nu_1}\partial x_2^{\nu_2}\cdots\partial x_n^{\nu_n}}\right|_{X=\overline{X}},$$