AN UPWIND DIFFERENCE SCHEME AND ITS CORRESPONDING BOUNDARY SCHEME*

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Abstract

An upwind difference scheme was given by the author in [5] for the numerical solution of steady-state problems. The present work studies this upwind scheme and its corresponding boundary scheme for the numerical solution of unsteady problems. For interior points the difference equations are approximations of the characteristic relations; for boundary points the difference equatons are approximations of the characteristic relations corresponding to the outgoing characteristics and the "non-reflecting" boundary conditions. Calculation of a Riemann problem in a finite computational region yields promising numerical results.

1. Introduction

Upwind difference schemes have always played important roles in the field of numerical solution of hyperbolic partial differential equations. We have, amongst the widely-used ones, the Courant, Isaacson, and Rees scheme [1], and the Godunov scheme, which is equivilent to an upwind scheme in a certain sense, see [2]. We also have [10], [3], [4], and many others. The author presented an upwind scheme in [5] for the numerical solution of steady-state problems. Its special feature is that the viscosity term concerned has effect in the unsteady process——it speeds up convergence; it has effect in the steady-state only in the shock region——it yields numerical shocks with at most one point of transition, but it does not influence the solution in the smooth region. Actually, the influence of vis-cosity on the smooth part of the solution depends directly on the boundary conditions. With suitable boundary conditions, the method under description consists of embedding a steady-state first order difference problem into an unsteady, second order (in space) difference problem. In [6] the author discussed the boundary scheme corresponding to the given upwind scheme and extended the boundary scheme to two-dimensional steady-state problems. This boundary scheme approximates the desired characteristic relations, it is in conservational form in the steady-state, and its implementation is especially convenient with implicit schemes.

The present work is an attempt to apply these schemes to the numerical solution of unsteady problems. A discussion of the upwind scheme and the corresponding boundary scheme is given in § 2. We shall see that for interior points, the difference equations are approximations of the characteristic relations. For boundary points,

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the difference equations are approximations of the characteristic relations corresponding to the outgoing characteristics, i. e. the desired characteristic relations, and the "non-reflecting" boundary conditions. When the region of solution is infinite, it is necessary to reduce it to a finite region for numerical solution. Then the "non-reflecting" boundary conditions have important significance. In § 3 the numerical solution of a Riemann problem in a finite computational region with the schemes discussed in § 2 is given. We shall see that the shock width is not large and that even when the rarefaction wave and the shock wave go out of the computational region, there are no apparent reflections from the boundaries.

2. An Upwind Scheme and Its Boundary Scheme

Consider the hyperbolic system

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}=0, \tag{1}$$

where U and F are p-dimensional vectors, $F=F(U)$. Let

$$A=\frac{\partial F}{\partial U},$$

then (1) can also be written as

$$\frac{\partial U}{\partial t}+A\frac{\partial U}{\partial x}=0.$$

Suppose $\lambda_1, \lambda_2, \cdots, \lambda_p$ are the eigenvalues of A, and $r_1, r_2, \cdots, r_p$ the corresponding right eigenvectors, then

$$A=R\Lambda R^{-1},\quad \Lambda=\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_p\end{pmatrix},$$

where R is the matrix $(r_1, r_2, \cdots, r_p)$. Let $L=R^{-1}$, then

$$L=\begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_p\end{pmatrix},$$

here l_i is the left eigenvector corresponding to λ_i. The characteristic normal form of (1) is

$$L\frac{\partial U}{\partial t}+\Lambda L\frac{\partial U}{\partial x}=0, \tag{2}$$

these are also called the characteristic relations.

The explicit form of the upwind scheme given in [5] is

$$\frac{U_j^{n+1}-U_j^n}{\Delta t}+\frac{F_{j+1}^n-F_{j-1}^n}{2\Delta x}=\frac{1}{2\Delta x}\left[\operatorname{sign} A_{j+\frac{1}{2}}^n\,(F_{j+1}^n-F_j^n)-\operatorname{sign} A_{j-\frac{1}{2}}^n(F_j^n-F_{j-1}^n)\right], \tag{3}$$