AN L_1 MINIMIZATION PROBLEM BY GENERALIZED RATIONAL FUNCTIONS*[1)]

SHI YING-GUANG(史应光)
(Computing Center, Academia Sinica)

Abstract

Let $P, Q\subset L_1(X, \Sigma, \mu)$ and $q(x)>0$ a. e. in X for all $q\in Q$. Define $R=\{p/q:p\in P, q\in Q\}$. In this paper we discuss an L_1 minimization problem of a nonnegative function $E(z, x)$, i. e. we wish to find a minimum of the functional $\phi(r)=\int_X qE(r, x)d\mu$ from $r=p/q\in R$. For such a problem we have established the complete characterizations of its minimum and of uniqueness of its minimum, when both P, Q are arbitrary convex subsets.

I. Introduction

Let (X, Σ, μ) be a σ-finite measure space and $L\equiv L_1(X, \Sigma, \mu)$ the linear normed space of all integrable functions on X with the norm

$$\|f\|=\int_X|f(x)|d\mu.$$

Assume that both P and Q are subsets in L and $q(x)>0$ almost everywhere in X for all $q\in Q$. Then we may construct the set of generalized rational functions

$$R=\{p/q:p\in P, q\in Q\}.$$

Suppose now that $E(z, x)$ is a nonnegative function from $(-\infty, \infty)\times X$ into $[0, \infty]$ such that $qE(r, \cdot)\in L$ for any element $r=p/q\in R$, where $E(r, \cdot)=E(r(\cdot),\cdot)$.

Our minimization problem then is to find an element $r_0=p_0/q_0\in R$ such that

$$\|q_0E(r_0, \cdot)\|=\inf_{r\in R}\|qE(r, \cdot)\|, \tag{1}$$

such an r_0 (if any) is called a minimum to E from R.

For a solution of the equation

$$\|E(r_0, \cdot)\|=\inf_{r\in R}\|E(r, \cdot)\|$$

we have not found, to the author's knowledge, its complete characterization and the complete characterization of its uniqueness. For a solution of equation (1), however, we can give all of them, provided that both P and Q are arbitrary convex subsets.

The minimization problem includes as special cases a number of ordinary and simultaneous approximation problems, such as

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$$E(z, x) = |f(x) - z|^s, \qquad 1 \leqslant s < \infty,$$
$$E(z, x) = \sum_i |f_i(x) - z|,$$
$$E(z, x) = \max_i |f_i(x) - z|,$$

etc.

II. Main Results

Suppose both P and Q are convex subsets in L. For $r = p/q$, $r_0 = p_0/q_0 \in R$ and $t \in [0, 1]$ write

$$p_t = p_0 + t(p - p_0),$$
$$q_t = q_0 + t(q - q_0),$$
$$r_t = p_t/q_t.$$

Our main results require several lemmas.

Lemma 1. *Let $f(x)$ be a convex function. Then for any $r = p/q$, $r_0 = p_0/q_0 \in R$*

$$\phi(t) \equiv (q_t f(r_t) - q_0 f(r_0))/t$$

is increasing with respect to t in $(0, 1]$.

Proof. Let $t \in (0, 1]$. Since

$$\phi(t) = \frac{q_t f(r_t) - q_0 f(r_0)}{t} = \frac{q_t(f(r_t) - f(r_0))}{t} + \frac{(q_t - q_0) f(r_0)}{t}$$
$$= q_t \cdot \frac{f(r_t) - f(r_0)}{r_t - r_0} \cdot \frac{r_t - r_0}{t} + (q - q_0) f(r_0)$$
$$= q(r - r_0) \cdot \frac{f(r_t) - f(r_0)}{r_t - r_0} + (q - q_0) f(r_0)$$

and

$$r_t - r_0 = \frac{tq}{q_0 + t(q - q_0)}(r - r_0),$$

for fixed x if $r(x) - r_0(x) > (<) 0$, $r_t(x) - r_0(x) > (<) 0$ and $r_t(x)$ is increasing (decreasing) with respect to t, which by the convexity of f implies that $(f(r_t(x)) - f(r_0(x)))/(r_t(x) - r_0(x))$ is increasing (decreasing) with respect to t [2, p. 6]. Thus in both the cases $\phi(t)$ is increasing with respect to t.

From $\phi(t) \leqslant \phi(1)$, $t \in (0, 1]$, we obtain the following lemma.

Lemma 2. *Let $f(x)$ be a convex function. Then for any $r = p/q$, $r_0 = p_0/q_0 \in R$*

$$(q_t f(r_t) - q_0 f(r_0))/t \leqslant q f(r) - q_0 f(r_0), \qquad t \in (0, 1]. \tag{2}$$

In order to state the following basic lemma we need to generalize the notion of the directional derivative to be applicable to our case. To this end for $r_i = p_i/q_i \in R$, $i = 0, 1, 2$, define

$$e(r_0, x; r_1, r_2) = \lim_{t \to 0+} \left[(q_0 + t(q_1 - q_2)) \, E\left(\frac{p_0 + t(p_1 - p_2)}{q_0 + t(q_1 - q_2)}, x\right) - q_0 E(r_0, x) \right] \Big/ t$$

if the limit exists. Hence

$$e(r_0, x; r, r_0) = \lim_{t \to 0+} (q_t E(r_t, x) - q_0 E(r_0, x))/t.$$

Lemma 3. *Let P and Q be convex sets in L. Suppose that $E(z, x)$ is convex with respect to z for each $x \in X$. Then for any $r = p/q$, $r_0 = p_0/q_0 \in R$*