Journal of Computational Mathematics
Vol.32, No.2, 2014, 152–168.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1310-FE3

A TRIANGULAR FINITE VOLUME ELEMENT METHOD FOR A SEMILINEAR ELLIPTIC EQUATION*

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Abstract

In this paper we extend the idea of interpolated coefficients for a semilinear problem to the triangular finite volume element method. We first introduce triangular finite volume element method with interpolated coefficients for a boundary value problem of semilinear elliptic equation. We then derive convergence estimate in H^1-norm, L^2-norm and L^∞-norm, respectively. Finally an example is given to illustrate the effectiveness of the proposed method.

Mathematics subject classification: 49J20, 65N30.
Key words: Semilinear elliptic equation, Triangulation, Finite volume element with interpolated coefficients.

1. Introduction

The finite volume element method is a discretization technique for partial differential equations, especially for those that arise from physical laws including mass, momentum, and energy. The finite volume element method uses a volume integral formulation of the differential equation with a finite partitioning set of volume to discretize the equation, then restricts the admissible functions to a linear finite element space to discretize the solution [2, 5–7, 19, 20, 22, 23, 25, 26, 29, 30, 33–36, 41]. The method has been widely used in computational fluid mechanics as it preserves the mass conservation. As far as the method is concerned, it is identical to the special case of the generalized difference method or GDM proposed by Li-Chen-Wu [29].

Many works have been devoted to the analysis of finite element methods. see, e.g., [11-18]. For semi-linear problems, the finite element method with interpolated coefficients is an economic and graceful method. This method was introduced and analyzed for semilinear parabolic problems in Zlamal [42]. Later Larsson-Thomee-Zhang [27] studied the semidiscrete linear triangular finite element with interpolated coefficients and Chen-Larsson-Zhang [10] derived almost optimal order convergence on piecewise uniform triangular meshes by the superconvergence techniques. Xiong-Chen studied superconvergence of triangular quadratic finite element and superconvergence of rectangular finite element for semilinear elliptic problem, respectively, and illustrated the effectiveness of the proposed method in some examples [37–39]. Recently Xiong-Chen first put the interpolation idea into the finite volume element method and studied the finite volume element with interpolated coefficients of the two-point boundary problem [40].

* Received November 28, 2012 / Revised version received July 19, 2013 / Accepted October 9, 2013 / Published online March 31, 2014 /

Li [28] considered the finite volume element method for a nonlinear elliptic problem and obtained the error estimate in H^1-norm. Chatzipantelidis-Ginting-Lazarov [8] studied the finite volume element method for a nonlinear elliptic problem, and established the error estimates in H^1-norm, L^2-norm and L^∞-norm. Bi [3] obtained the H^1 and $W^{1,\infty}$ superconvergence estimates between the solution of the finite volume element method and that of the finite element method for a nonlinear elliptic problem. In this paper, we shall put the excellent interpolating coefficients idea into the finite volume element method on triangular mesh for a semilinear elliptic equation.

We shall denote Sobolev space and its norm by $W^{m,p}(\Omega)$ and $\|\cdot\|_{m,p}$, respectively [1]. If $p=2$, simply use $H^m(\cdot)$ and $\|\cdot\|_m$ and $\|\cdot\|=\|\cdot\|_0$ is L^2-norm. Further we shall denote by p' the adjoint of p, i.e., $\frac{1}{p}+\frac{1}{p'}=1,\ p\geq 1$. We shall assume that the exact solution u is sufficiently smooth for our purpose. The constants C, C_1, C_2, etc. are generic in the paper.

The rest of the paper is organized as follow. First we will introduce the triangular finite volume element method with interpolated coefficients in Section 2 and give preliminaries and some lemmas in Section 3. Next we derive optimal order H^1-norm, L^2-norm and L^∞-norm estimates, respectively, in Section 4. Finally the theoretical results are tested by a numerical example in Section 5.

2. Finite Volume Element Method with Interpolated Coefficients

Let $\Omega\subset\mathbb{R}^2$ be a bounded polygonal domain. Consider the second-order semilinear elliptic boundary value problem:

$$\begin{cases} -\frac{\partial}{\partial x}\Big(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\Big)-\frac{\partial}{\partial y}\Big(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\Big)+f(u)=g, \quad \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where the coefficients $a_{ij}(x,y)(i,j=1,2)$ are sufficiently smooth functions satisfying the elliptic condition, i.e., there exists a constant $C>0$ such that

$$\sum_{i,j=1}^{2} a_{ij}(x,y)\xi_i\xi_j \geq C(\xi_1^2+\xi_2^2),$$

holds for any real vector $(\xi_1,\xi_2)\in\mathbb{R}^2$ and $(x,y)\in\overline{\Omega}$. It is also assumed that $f'(s)>0$ for $s\in(-\infty,+\infty)$ and $f''(s)$ is continuous with respect to s.

Let $V\subset\Omega$ be any control volume with piecewise smooth boundary ∂V. Integrate (2.1) over control volume V, then by the Green's formula, the conservative integral of (2.1) reads, finding u, such that

$$-\int_{\partial V} W^{(1)}\mathrm{d}y+\int_{\partial V} W^{(2)}\mathrm{d}x+\int_V f(u)\mathrm{d}x\mathrm{d}y=\int_V g\mathrm{d}x\mathrm{d}y, \quad V\subset\Omega, \tag{2.2}$$

where

$$W^{(i)}=a_{i1}\frac{\partial u}{\partial x}+a_{i2}\frac{\partial u}{\partial y}, \quad i=1,2.$$

In this paper, we shall consider triangular partition of Ω and piecewise triangle linear interpolation with interpolated coefficients, for u.

Give a quasi-uniform triangulation $\mathcal{J}_h$ for Ω with $h=\max h_K$, where h_K is the diameter of the triangle $K\in\mathcal{J}_h$. All control volumes are constructed in the following way. Let Q_K be