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ALTERNATELY LINEARIZED IMPLICIT ITERATION METHODS FOR SOLVING QUADRATIC MATRIX EQUATIONS*

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Abstract

A numerical solution of the quadratic matrix equations associated with a nonsingular M-matrix by using the alternately linearized implicit iteration method is considered. An iteration method for computing a nonsingular M-matrix solution of the quadratic matrix equations is developed, and its corresponding theory is given. Some numerical examples are provided to show the efficiency of the new method.

Mathematics subject classification: 65F10.
Key words: Quadratic matrix equation, Alternately iteration, M-matrix, Matrix transformation.

1. Introduction

We consider the numerical solution of the following matrix equation

$$X^2 - BX - C = 0, \tag{1.1}$$

where $B, C, X \in R^{n\times n}$, all off-diagonal elements of B are nonnegative and C is a nonsingular M-matrix. The nonlinear matrix equation has numerous applications in control theory, signal processing and so on [1,2]. Some methods [3-6] have been developed extensively for solving the matrix equation. By simply transforming the quadratic matrix equation into an equivalent fixed-point equation, Bai et al. [3] constructed a successive approximation method and a Newton method based on the fixed-point equation. Higham and Kim [4] incorporated exact line searches into Newton method to solve the quadratic matrix equation.

Recently, Bai, Guo and Xu [7] proposed an alternately linearized implicit (ALI) iteration method for computing the minimal nonnegative solution of the algebraic Riccati equations (AREs). This method is more feasible and effective than the other methods. Applying this method, in this paper we propose a new numerical method for solving the quadratic matrix equation (1.1). The method for computing a nonsingular M-matrix solution of the quadratic matrix equation (1.1) is developed and its corresponding theory is given. Some numerical examples are provided to show the efficiency of the new method.

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We first define some notations and introduce several basic results. For any matrices $A, B \in R^{n\times n}$, we write $A \geq B (A > B)$ if $a_{ij} \geq b_{ij} (a_{ij} > b_{ij})$ for all i, j. A real square matrix A is called a Z -matrix if all its off-diagonal elements are nonpositive. It follows that any Z -matrix A can be written as the form $A = sI - B$, with s a positive real and B a nonnegative matrix. A Z-matrix A is called an M-matrix if $s \geq \rho(B)$, where $\rho(B)$ denotes the spectral radius of B. It is called a singular M-matrix if $s = \rho(B)$; It is called a nonsingular M-matrix if $s > \rho(B)$. $\|A\|$ denotes the Frobenius norm of a matrix A.

Lemma 1.1. ([8]) *For a Z -matrix A , the following statements are equivalent:*

(1) *A is a nonsingular M-matrix;*

(2) $A^{-1} \geq 0$*;*

(3) $Av > 0$ *for some vector* $v > 0$*.*

Lemma 1.2. ([7]) *Let*

$$K = \begin{bmatrix} D & -C \\ -B & A \end{bmatrix}.$$

If K is a nonsingular M-matrix, then the algebraic Riccati equation (ARE)

$$\Re(X) = XCX - XD - AX + B = 0 \tag{1.2}$$

has a minimal nonnegative solution S , where A, B, C and D are real matrices of sizes $m \times m, m \times n, n \times m$ and $n \times n$, respectively.

Bai et al. [7] established a class of alternately linearized implicit (ALI) iteration methods for computing the minimal nonnegative solutions of the ARE (1.2) by the following algorithm.

Algorithm 1.1. ([7]) (The ALI iteration method)

1) Set $X_0 = 0 \in R^{m\times n}$.

2) For $k = 0, 1, \cdots$, until $\{X_k\}$ convergence, compute $\{X_{k+1}\}$ from $\{X_k\}$ by solving the following two systems of linear matrix equations:

$$\begin{cases} X_{k+\frac{1}{2}} (\alpha I + (D - CX_k)) = (\alpha I - A) X_k + B, \\ \left(\alpha I + \left(A - X_{k+\frac{1}{2}} C\right)\right) X_{k+1} = X_{k+\frac{1}{2}} (\alpha I - D) + B, \end{cases}$$

where $\alpha > 0$ is a given iteration parameter.

The ALI iteration method is better than both the Newton iteration method and the fixed-point iteration method.

We consider the numerical solution of the quadratic matrix equation (1.1) associated with a nonsingular M-matrix by using the ALI iteration method. The paper is organized as follows. First, the quadratic matrix equation (1.1) is transformed into the algebraic Riccati equations. Second, a numerical method for computing an M-matrix solution of the quadratic matrix equation is proposed in Section 2. Then, some numerical examples are provided in Section 3. Finally, conclusions are given in Section 4.