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Global Strong Solution to the 3D Incompressible Navierv-Stokes Equations with General Initial Data

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Abstract. We study the existence of global strong solution to an initial–boundary value (or initial value) problem for the 3D nonhomogeneous incompressible Navier-Stokes equations. In this study, the initial density is suitably small (or the viscosity coefficient suitably large) and the initial vacuum is allowed. Results show that the unique solution of the Navier-Stokes equations can be found.

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1 Introduction

The motion of a nonhomogeneous incompressible viscous fluid in a domain Ω of $\mathbb{R}^3$ is governed by the Navier-Stokes equations

$$\begin{cases} \rho_t+\mathrm{div}(\rho u)=0, \\ (\rho u)_t+\mathrm{div}(\rho u\otimes u)-\mu\triangle u+\nabla P=\rho f, \\ \mathrm{div} u=0, \end{cases} \tag{1.1}$$

the initial and boundary conditions (1.1) with the following conditions:

$$\begin{cases} (\rho,u)|_{t=0}=(\rho_0,u_0), \text{ in } \Omega, \;\; u=0 \text{ on } \partial\Omega\times(0,T), \\ u(x,t)\to 0, \; \rho(x,t)\to 0, \text{ as } |x|\to\infty, \; (x,t)\in\Omega\times(0,T). \end{cases} \tag{1.2}$$

Here we denote the unknown density, velocity and pressure fields of the fluid by ρ, u and P , respectively. f is a given external force driving the motion. Ω is either a bounded domain in $\mathbb{R}^3$ with smooth boundary or the whole space $\mathbb{R}^3$.

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It is interesting to studing the regularity criterion for strong solution of (1.1). Many people devote to researching these kind of results. In particular, Kim [1] proved that if T^* was the blowup time of a local strong solution, then

$$\int_0^{T^*} \|u(t)\|_{L^r_w}^s dt = \infty, \text{ for any } (r,s) \text{ with } \frac{2}{s}+\frac{3}{r}=1, \; 3<r<\infty,$$

where L^r_w denoted the weak L^r-space. In [1], Kim also proved that the unique strong solution existed globally when $\|\nabla u_0\|_{L^2}$ was small enough.

For the case the initial density is away from zero, the nonhomogeneous equations (1.1) have been studied by many people, see [2–4] and their references therein. In these papers, the authors proved the existence and uniqueness of the local strong solution for general initial data and they also got global well-posedness results for small solutions in 3D (or higher dimensional) space, while for 2D space they established the existence of large strong solutions. In [5–7], the authors obtained the global well-posedness results for initial data belonging to certain scale invariant space.

For the case the initial density vanished in some sets, DiPerna and Lions [8,9] proved the global existence of weak solutions to (1.1) in any dimensional space. In [10, 12], the authors proved the existence of local smooth or weak solutions.

In this paper, base on Kim's work, we are interested in the existence of global strong solution with general initial data. The main result of this paper can be stated as follows:

Theorem 1.1. *Assume that* (ρ_0,u_0,f) *satisfies*

$$\begin{aligned} &0\le\rho_0\in L^{\frac{3}{2}}\cap H^2, \quad u_0\in D^1_{0,\sigma}\cap D^2, \\ &f\in L^2(0,\infty;H^1) \text{ and } f_t\in L^2(0,\infty;L^2) \end{aligned} \tag{1.3}$$

and the compatibility condition

$$-\mu\triangle u_0+\nabla P_0=\sqrt{\rho_0}g, \quad divu_0=0 \text{ in } \Omega, \tag{1.4}$$

for some $(P_0,g)\in D^1\times L^2$*. Moreover, for some given positive numbers* M *and* $\bar\rho$*, the initial data* (ρ_0,u_0) *satisfy*

$$0\le\inf\rho_0\le\sup\rho_0\le\bar\rho, \quad \|\nabla u_0\|_{L^2}^2\le M.$$

If $\bar\rho$ *is small or* μ *is large, there exists a unique global strong solution to the nonhomogeneous Navier-Stokes equations* (1.1)-(1.2) *for any* $T<\infty$.

Throughout this paper, we denote

$$\int f dx=\int_\Omega f dx,$$

$1<r<\infty$, k is a positive constant, the standard Sobolev space is described as follows:

$$\begin{cases} L^r=L^r(\Omega),\, D^{k,r}=\{u\in L^1_{\text{loc}}(\Omega)|\ \|\nabla^k u\|_{L^r}<\infty\},\, \|u\|_{D^{k,r}}\triangleq\|\nabla^k u\|_{L^r}, \\ W^{k,r}=L^r\cap D^{k,r},\, H^k=W^{k,2},\, D^k=D^{k,2}, \\ D^1_0=\{u\in L^6|\ \|\nabla u\|_{L^2}<\infty, \text{ and } u=0 \text{ on } \partial\Omega\}, \\ H^1_0=L^2\cap D^1_0,\, D^1_{0,\sigma}=\{u\in D^1_0|\ \text{div}u=0 \text{ in } \Omega\}. \end{cases}$$