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Existence and Stability of Solitary Waves of An M-Coupled Nonlinear Schrödinger System

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Abstract. In this paper, the existence and stability results for ground state solutions of an m-coupled nonlinear Schrödinger system

$$i\frac{\partial}{\partial t}u_j+\frac{\partial^2}{\partial x^2}u_j+\sum_{i=1}^{m}b_{ij}|u_i|^p|u_j|^{p-2}u_j=0,$$

are established, where $2\leq m$, $2\leq p<3$ and u_j are complex-valued functions of $(x,t)\in\mathbb{R}^2$, $j=1,\cdots,m$ and b_{ij} are positive constants satisfying $b_{ij}=b_{ji}$. In contrast with other methods used before to establish existence and stability of solitary wave solutions where the constraints of the variational minimization problem are related to one another, our approach here characterizes ground state solutions as minimizers of an energy functional subject to independent constraints. The set of minimizers is shown to be orbitally stable and further information about the structure of the set is given in certain cases.

AMS subject classifications: 35A15, 35B35, 35Q35.

Key words: Orbital stability, coupled NLS systems, vector solutions, ground-state solutions.

1 Introduction

The nonlinear Schrödinger(NLS) equation

$$iu_t+u_{xx}\pm|u|^{p-2}u=0, \tag{1.1}$$

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where $2 < p$ and u is a complex function of $(x,t) \in \mathbb{R}^2$ arises in several applications. The equation describes evolution of small amplitude, slowly varying wave packets in a nonlinear media [3]. Indeed, it has been derived in such diverse fields as deep water waves [23], plasma physics [24], nonlinear optical fibers [9, 10], magneto-static spin waves [25], to name a few. The coupled nonlinear Schrödinger (CNLS) system

$$i\frac{\partial}{\partial t}u_j+\frac{\partial^2}{\partial x^2}u_j+\sum_{i=1}^{m}b_{ij}|u_i|^p|u_j|^{p-2}u_j=0, \tag{1.2}$$

where $2 \le p < 3$, $m \ge 2$ and u_j are complex-valued functions of $(x,t) \in \mathbb{R}^2$, $j=1,2,\cdots,m$ and $b_{ij} \in \mathbb{R}$, arises physically under conditions similar to those described by (1.1) when there are m-wave trains moving with nearly the same group velocities [20,22]. The CNLS system also models physical systems whose fields have more than one components; for example, in optical fibers and waveguides, the propagating electric field has two components that are transverse to the direction of propagation. These types of systems also arise from physical models in nonlinear optics and in Bose-Einstein condensates for multi-species condensates (i.e., [15, 21] and references therein). Readers are referred to the works [3, 9, 10, 15, 21, 23, 24] for the derivation as well as applications of the system (1.2). With coupling effects in the system, some new features of the solutions structure arise that do not exist in the single equation (1.1).

Notation. For $1 \le p \le \infty$, we denote by $L^p = L^p(\mathbb{R})$ the space of all complex-valued measurable functions f on $\mathbb{R}$ for which the norm $\|f\|_p=(\int_{\mathbb{R}}|f|^p dx)^{\frac{1}{p}}$ is finite for $1\le p<\infty$, and $\|f\|_\infty$ is the essential supremum of $|f|$ on $\mathbb{R}$. The space $H^1_{\mathbb{C}}(\mathbb{R})$ is the usual Sobolev space consisting of measurable functions such that both f and f_x are in L^2 and we define the space X_j to be the j-times Cartesian product $X_j=H^1_{\mathbb{C}}(\mathbb{R})\times H^1_{\mathbb{C}}(\mathbb{R})\times\cdots\times H^1_{\mathbb{C}}(\mathbb{R})$. If $T>0$ and Y is any Banach space, we denote by $\mathcal{C}([0;T],Y)$ the Banach space of continuous maps $f:[0,T]\longrightarrow Y$, with norms given by $\|f\|_{\mathcal{C}([0;T],Y)}=\sup_{[0,T]}\|f(t)\|_Y$.

Review. Global well-posedness for the system (1.2) follows from [6] (see also [16]). Precisely, it was proved that for any initial data $(u_1(x,0),u_2(x,0),\cdots,u_m(x,0))\in X_m$, there exists a unique solution $(u_1(x,t),u_2(x,t),\cdots,u_m(x,t))$ of (1.2) in $\mathcal{C}(\mathbb{R};X_m)$ emanating from $(u_1(x,0),u_2(x,0),\cdots, u_m(x,0))$, and $(u_1(x,t),u_2(x,t),\cdots,u_m(x,t))$ satisfies

$$\begin{aligned}&E(u_1(\cdot,t),u_2(\cdot,t),\cdots,u_m(\cdot,t))=E(u_1(\cdot,0),u_2(\cdot,0),\cdots,u_m(\cdot,0)),\\&Q(u_j(\cdot,t))=Q(u_j(\cdot,0)),\end{aligned}$$

where E and Q are the following conserved quantities

$$\begin{aligned}&E(u_1,u_2,\cdots,u_m)=\int_{\mathbb{R}}\left(\sum_{j=1}^{m}|\frac{\partial}{\partial x}u_j(x,t)|^2-\frac{1}{p}\sum_{i,j=1}^{m}b_{ij}|u_i(x,t)|^p|u_j(x,t)|^p\right)dx,\\&Q(u_j)=\int_{\mathbb{R}}|u_j(x,t)|^2dx,\end{aligned}$$