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Finite Difference/Collocation Method for Two-Dimensional Sub-Diffusion Equation with Generalized Time Fractional Derivative

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Abstract. In this paper, we propose a finite difference/collocation method for two-dimensional time fractional diffusion equation with generalized fractional operator. The main purpose of this paper is to design a high order numerical scheme for the new generalized time fractional diffusion equation. First, a finite difference approximation formula is derived for the generalized time fractional derivative, which is verified with order $2-\alpha$ $(0<\alpha<1)$. Then, collocation method is introduced for the two-dimensional space approximation. Unconditional stability of the scheme is proved. To make the method more efficient, the alternating direction implicit method is introduced to reduce the computational cost. At last, numerical experiments are carried out to verify the effectiveness of the scheme.

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1 Introduction

The time fractional diffusion equation (TFDE) is obtained by replacing the first-order time derivative with fractional derivative of order $\alpha\,(0<\alpha<1)$. This model equation governs the evolution for the probability density function that describes anomalously diffusing particles. Examples for sub-diffusive transport include turbulent flow, chaotic dynamics charge transport in amorphous semiconductors [1, 2], NMR diffusometry in disordered

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materials [3], and dynamics of a bead in polymer network [4]. Literatures on TFDE are mainly based on Riemann-Liouville derivative or Caputo derivative. In addition to that, Erdélyi-Kober fractional derivative and other generalized fractional derivatives are also important for some special problems. In [5], Sandev et al. investigated a kind of diffusion equation with a Hilfer-generalized Riemann-Liouville time fractional derivative. By using methods of separation of variables and Laplace transform, the solution of the fractional diffusion equation is obtained in terms of Mittag-Leffler-type functions and Foxs H-function. In [6], the solution of space-time fractional diffusion equations with a generalized Riemann-Liouville time fractional derivative and Riesz-Feller space fractional derivative is studied. The Laplace and Fourier transform methods are applied to solve the proposed fractional diffusion equation and the solutions are expressed by using the Mittag-Leffler functions and the Fox H-function. In [7], Pagnini proposed a new fractional diffusion model for generalized grey Brownian motion (ggBm) driven by a fractional integral equation in the sense of Erdélyi-Kober. The ggBm is a parametric class of stochastic processes that provides models for both fast and slow anomalous diffusion. For this reason, this new model is called Erdélyi-Kober fractional diffusion. A linear space-time fractional reaction-diffusion equation with composite time fractional derivative and Riesz-Feller space fractional derivative is investigated in [8] by Garg. Laplace and Fourier transforms is applied to obtain its solution. The fractional derivatives used in these papers are different with the classical Riemann-Liouville or Caputo fractional derivatives. In addition to these, there are still other definitions proposed for some special applications.

The notion "generalized operator of fractional integration" appeared first in the papers of the jubilarian professor S. L. Kalla in the years 1969–1979 [9, 10]. The basic ideas of the generalized fractional calculus is surveyed in [11, 12] and further generalizations of fractional integrals and derivatives is presented by Agrawal in [13]. Due to the special formulation, almost all known fractional integrals and derivatives, such as Hadamard operator and the Erdélyi-Kober operator, in various areas of analysis happened to fall in the framework of this generalized fractional calculus. In this paper, we consider the time fractional diffusion equation with generalized fractional derivative proposed by Agarwal. Analytical solutions of fractional differential equations are generally expressed using Mittag-Leffler-type functions and Fox H-functions, which are difficult to evaluation. Thus numerical method is a powerful tool for solving fractional differential equations. There are already abundant of papers working on numerical methods for time fractional diffusion with classical Caputo or Riemann-Liouville derivatives, such as [14, 15, 17–21, 26], to name a few. But there are still very few papers on time diffusion equations with generalized fractional derivatives. In [6], numerical scheme and Grünwald-Letnikov approximation is used to solve the space-time fractional diffusion equation with a generalized Riemann-Liouville time fractional derivative and Riesz-Feller space fractional derivative. In 2013, Xu [22] propose a numerical scheme for one-dimensional time fractional advection-diffusion equations with generalized fractional derivative. Later, a similar numerical method and an analytical solution are proposed for