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Modified Galerkin Method for Derivative Dependent Fredholm–Hammerstein Integral Equations of Second Kind

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Abstract. In this paper, we consider modified Galerkin and iterated modified Galerkin methods for solving a class of two point boundary value problems. The methods are applied after constructing the equivalent derivative dependent Fredholm-Hammerstein integral equations to the boundary value problem. Existence and convergence of the approximate solutions to the actual solution is discussed and the rates of convergence are obtained. Superconvergence results for the approximate and iterated approximate solutions of piecewise polynomial based modified Galerkin method in infinity norm are given. We have also established that iterated modified Galerkin approximation improves over the modified Galerkin solution. Numerical examples are presented to illustrate the theoretical results.

AMS subject classifications: 45B05, 45G10, 65R20

Key words: Fredholm integral equations, Green's kernel, modified Galerkin method, piecewise polynomial, superconvergence rates.

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1 Introduction

Consider the following two-point boundary value problem

$$(\vartheta'(t))' = \phi(t,\vartheta(t),\vartheta'(t)), \tag{1.1}$$

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subject to the boundary conditions

$$\vartheta(0)=\alpha_1, \quad \beta_1\vartheta(1)+\gamma_1\vartheta'(1)=\eta_1.$$

Frequently, different phenomena in scientific fields, including mechanics, optimization, communication theory, fluid mechanics, electricity, magnetism, and many other applied science problems are reduced to solve the boundary value problem (1.1). The numerical treatment of the above boundary value problems has always been far from trivial. The following integral equations arise as reformulation of the above type singular two-point boundary value problem (1.1)

$$\vartheta(t)=\alpha_1+\frac{(\eta_1-\alpha_1\beta_1)t}{\beta_1+\gamma_1}+\int_0^1 \kappa(t,\chi)\phi(\chi,\vartheta(\chi),\vartheta'(\chi))d\chi, \quad 0\le t\le 1, \tag{1.2}$$

where $\kappa(t,\chi)$ is given by

$$\kappa(t,\chi)=\begin{cases} t\Big(1-\dfrac{\beta_1\chi}{\beta_1+\gamma_1}\Big), & 0\le t\le \chi, \\ \chi\Big(1-\dfrac{\beta_1 t}{\beta_1+\gamma_1}\Big), & \chi\le t\le 1. \end{cases}$$

The main difficulty of (1.1) is that the singularity behavior occurs at $t=0$. With the use of important properties of Green's functions, it would be easier to handle these equations after constructing the equivalent nonlinear Fredholm integral equations. The same is mentioned in [27], where authors discussed numerical solvability of the similar kind of singular two-point boundary value problem after reformulating them into a nonlinear Fredholm integral equation with Green's kernel. Also, with this reformultaion, the higher order derivative approximation for (1.1) can be avoided, which is computationally very much favorable. In the last few years, effective methods such as decomposition method, the Adomian decomposition method, and the modified decomposition method etc. are developed for numerically solving different types of boundary value problems and associated integral equations (see [1,6,7,15,16,27]). In attempt of improving the accuracy of the approximate solutions, projection methods are used to solve Fredholm integral equations. Several results on different projection methods to solve nonlinear Fredholm integral equations can be found in literature (see [11,12,17,19,21,22,25,26]). Classical projection methods such as Galerkin, collocation methods for Fredholm Hammerstein integral equations with smooth as well as weakly singular kernels were developed and superconvergence was obtained by several authors (see [9,11,12,17–19,21,22,26]) etc.). Piecewise polynomial based Galerkin method is applied to investigate the approximate solutions of nonlinear Fredholm-Hammerstein integral equations with smooth kernels in [9]. Authors developed projection and iterated projection methods to solve nonlinear Fredholm-integral equation with particular classes of kernels having singularity (see [3–5]).

In literature, many attempts have been made to improve the accuracy of numerical solutions of different integral equations using projection methods. In [20], authors created the modified projection method and showed that under the same assumptions of classical projection methods, the proposed modified projection methods exhibit superconvergence results over iterated projection methods. Also, authors had shown that, the Computational complexities are almost same in modified projection methods as classical projection methods. After that, modified projection methods have been applied in several papers for solving nonlinear type integral equations with smooth kernels (see [13,14] etc). Now, in [24], M. Mandal et. al. applied Galerkin and iterated Galerkin methods using piecewise polynomials to solve Eq. (1.2) and obtain the rate of convergence as $\mathcal{O}(h^p)$ in Galerkin and $\mathcal{O}(h^{p+p_2})$ in iterated Galerkin method, where h is the norm of the partition and $p=\min\{r_1,r+1\}$, $p_1=\min\{r_1,r+1,\gamma+2\}$, $p_2=\min\{r_1-1,r+1,\gamma+1\}$, where r is the degree of the piecewise polynomial of the finite dimensional approximation space, r_1 is the smoothness of the solution, and γ is such that $r_1\geq\gamma\geq-1$ and $\kappa(t,\chi)\in\mathcal{C}^{r_1}(0,t)\cap\mathcal{C}^{r_1}(t,1)\cap\mathcal{C}^{\gamma}(0,1)$. In this paper, aiming at the improvement of these convergence rates, we have applied modified Galerkin and iterated modified Galerkin methods to solve the nonlinear Fredholm integral equation of the type (1.2), and obtain the convergence rates $\mathcal{O}(h^{p+p_2})$ in modified Galerkin method and $\mathcal{O}(h^{p+2p_2})$ in iterated modified Galerkin method, respectively in uniform norm.

We organize this paper as follows. In Section 2, we analyze piecewise polynomial based modified Galerkin and iterated modified Galerkin methods to solve Eq. (1.2). In Section 3, we obtain superconvergence results for approximate solutions. In Section 4, we have validated the theoretical estimates with numerical examples. Throughout this paper, we assume that c is a generic constant.

2 Modified projection methods: derivative dependent Fredholm-Hammerstein integral equations with a Green's kernel

Let $\mathbb{X}=\mathcal{C}[0,1]$ and consider the following derivative dependent Fredholm-Hammerstein integral equation

$$\vartheta(t)=f(t)+\int_0^1\kappa(t,\chi)\phi(\chi,\vartheta(\chi),\vartheta'(\chi))d\chi,\ \ 0\leq t\leq 1, \tag{2.1}$$

with Green's function $\kappa(t,\chi)$

$$\kappa(t,\chi)=\begin{cases} t\Big(1-\dfrac{\beta_1\chi}{\beta_1+\gamma_1}\Big), & 0\leq t\leq\chi,\\ \chi\Big(1-\dfrac{\beta_1 t}{\beta_1+\gamma_1}\Big), & \chi\leq t\leq 1,\end{cases}$$

where the functions κ, f, and ϕ are known and ϑ is the unknown function to be determined.

We define the operator $(\mathcal{K}\phi)$ as follows:

$$(\mathcal{K}\phi)(\vartheta)(x)=\int_0^1 \kappa(x,s)\phi(s,\vartheta(s),\vartheta'(s))ds, \quad x\in[0,1]. \tag{2.2}$$

Then using (2.2), Eq. (2.1) can be written as

$$\vartheta(t)-(\mathcal{K}\phi)(\vartheta)(t)=f(t), \quad 0\le t\le 1. \tag{2.3}$$

For $t\in[0,1]$, we define

$$\kappa_t(\chi)=\kappa(t,\chi), \quad \ell_t(\chi)=\ell(t,\chi)=\frac{\partial\kappa}{\partial t}(t,\chi),$$

and

$$\kappa_{1t}(\chi)=\kappa_t(\chi), \qquad 0\le t\le\chi, \tag{2.4a}$$

$$\kappa_{2t}(\chi)=\kappa_t(\chi), \qquad \chi\le t\le 1. \tag{2.4b}$$

We assume that $0\le t\le 1$, $\kappa_{1t}\in\mathcal{C}^{r_1}[0,t]$, $\kappa_{2t}\in\mathcal{C}^{r_1}[t,1]$ and $\kappa(t,\chi)\in\mathcal{C}^{r_1}(0,t)\cap\mathcal{C}^{r_1}(t,1)\cap\mathcal{C}^{\gamma}(0,1)$ and

$$\ell(t,\chi)=\frac{\partial\kappa}{\partial t}(t,\chi)\in\mathcal{C}^{r_1-1}(0,t)\cap\mathcal{C}^{r_1-1}(t,1)\cap\mathcal{C}^{\gamma-1}(0,1), \quad \text{with } r_1\ge 1 \quad \text{and } r_1\ge\gamma\ge -1.$$

We assume that $f\in\mathbf{C}^{r_1}[0,1]$. Consequently, from Theorem 4.1 and Corollary 4.2 of [5], it follows that $\vartheta\in\mathbf{C}^{r_1}[0,1]$. We let

$$\|v\|_{r_1,\infty}=\max\{\|v^{(i)}\|_\infty : 0\le i\le r_1\},$$

where $v^{(i)}$ denote the i-th derivative of v.

Next we take the following assumptions on f, $\kappa(t,\chi)$ and $\phi(\cdot,\vartheta(\cdot),\vartheta'(\cdot))$:

(i) $f\in\mathcal{C}^{r_1}[0,1]$.

(ii) $A_1=\sup\limits_{t,\chi\in[0,1]}|\kappa(t,\chi)|<\infty$, $A_2=\sup\limits_{t,\chi\in[0,1]}|\ell(t,\chi)|<\infty$.

(iii) The nonlinear function $\phi(\chi,\vartheta,\vartheta')$ is Lipschitz continuous in ϑ and ϑ', i.e., for any $\vartheta_1,\vartheta_2,\vartheta'_1,\vartheta'_2\in\mathbb{X}$, $\exists c_1>0$, such that

$$|\phi(\chi,\vartheta_1,\vartheta'_1)-\phi(\chi,\vartheta_2,\vartheta'_2)|\le c_1\{|\vartheta_1(\chi)-\vartheta_2(\chi)|+|\vartheta'_1(\chi)-\vartheta'_2(\chi)|\}, \quad \forall\chi\in[0,1].$$

(iv) The partial derivatives $\phi^{(0,1,0)}(\chi,\vartheta,\vartheta')$, $\phi^{(0,0,1)}(\chi,\vartheta,\vartheta')$ of ϕ w.r.t the second and third variables exists and are Lipschitz continuous in ϑ and ϑ', i.e., for any $\vartheta_1,\vartheta_2,\vartheta'_1,\vartheta'_2\in\mathbb{X}$, $\exists c_2,c_3>0$, such that

$$\begin{aligned}&|\phi^{(0,1,0)}(\chi,\vartheta_1,\vartheta'_1)-\phi^{(0,1,0)}(\chi,\vartheta_2,\vartheta'_2)|\\ &\le c_2\{|\vartheta_1(\chi)-\vartheta_2(\chi)|+|\vartheta'_1(\chi)-\vartheta'_2(\chi)|\}, \qquad \forall\chi\in[0,1],\\ &|\phi^{(0,0,1)}(\chi,\vartheta_1,\vartheta'_1)-\phi^{(0,0,1)}(\chi,\vartheta_2,\vartheta'_2)|\\ &\le c_3\{|\vartheta_1(\chi)-\vartheta_2(\chi)|+|\vartheta'_1(\chi)-\vartheta'_2(\chi)|\}, \qquad \forall\chi\in[0,1],\end{aligned}$$

and $\phi^{(0,1,0)},\phi^{(0,0,1)}\in\mathcal{C}([0,1]\times\mathbb{X}\times\mathbb{X})$.

(v) We also assume that $Ac_1 < 1$, where $A = A_1 + A_2$.

We take

$$\mathcal{K}v(t) = \int_0^1 \kappa(t,\chi)v(\chi)d\chi, \tag{2.5}$$

and

$$\mathcal{L}v(t) = \frac{d}{dt}(\mathcal{K}v)(t) = \int_0^1 \ell(t,\chi)v(\chi)d\chi, \quad \text{where } \ell(t,\chi) = \frac{\partial\kappa}{\partial t}(t,\chi). \tag{2.6}$$

Note that $\mathcal{K}:\mathbb{X}\to\mathbb{X}$ and $\mathcal{L}:\mathbb{X}\to\mathbb{X}$ are compact operators and

$$\|\mathcal{K}\|_\infty \le A_1 \quad \text{and} \quad \|\mathcal{L}\|_\infty \le A_2. \tag{2.7}$$

Now we will rewrite Eq. (2.1) using the technique introduced by Kumar and Sloan [22]. To do this, we let

$$\varrho(\chi) = \phi(\chi,\vartheta(\chi),\vartheta'(\chi)), \quad \chi\in[0,1]. \tag{2.8}$$

Note that if $\phi(\cdot,\cdot,\cdot)\in\mathcal{C}^{r_1}([0,1]\times[0,1]\times[0,1])$, then using the chain rule for higher derivatives, we can say that $\varrho\in\mathcal{C}^{r_1}[0,1]$.

Using (2.8), we can write the solution ϑ of (2.1) satisfies the following

$$\vartheta(t) = f(t) + \int_0^1 \kappa(t,\chi)\varrho(\chi)d\chi, \quad 0\le t\le 1. \tag{2.9}$$

Hence using (2.5), Eq. (2.9) becomes

$$\vartheta = f + \mathcal{K}\varrho. \tag{2.10}$$

For our convenience, we consider a nonlinear operator $\Phi:\mathbb{X}\to\mathbb{X}$ defined by

$$\Phi(\vartheta)(\chi) = \phi(\chi,\vartheta(\chi),\vartheta'(\chi)). \tag{2.11}$$

Then using estimates (2.10) and (2.11), Eq. (2.8) can be written as

$$\varrho = \Phi(\vartheta) = \Phi(f+\mathcal{K}\varrho). \tag{2.12}$$

Letting $\mathcal{T}(v) := \Phi(f+\mathcal{K}v)$, $v\in\mathbb{X}$, Eq. (2.12) can be written as

$$\varrho = \mathcal{T}\varrho. \tag{2.13}$$

By our assumption $Ac_1 < 1$, $\mathcal{T}$ can be shown contraction mapping on $\mathcal{X}$ and hence by Banach contraction theorem, Eq. (2.13) has a unique solution ϱ_0 in $\mathbb{X}$.

To analyze the modified Galerkin method, we let the approximating subspaces

$$\mathbb{X}_h = \mathcal{P}_{r,\Delta} = \{\vartheta : \vartheta|_{(x_{i-1},x_i)} \in \mathcal{P}_r,\ 1\le i\le n\},$$

where $\mathcal{P}_r$ denote the space of (real) polynomials of degree $\le r$, where $r\ge 1$. For $g\in\mathcal{P}_{r,\Delta}$, if the value at x_i is defined by continuity, then $\mathcal{P}_{r,\Delta}\subset\mathcal{C}_\Delta$ and the projection $\mathcal{P}_h$ is defined from $\mathcal{C}_\Delta$ onto $\mathcal{P}_{r,\Delta}$ with $g=(g_1,g_2,\cdots,g_n)\to\mathcal{P}_h g = (\mathcal{P}g_1,\mathcal{P}g_2,\cdots,\mathcal{P}g_n)$, where $\mathcal{P}g_i$ is the orthogonal projection of $g_i\in\mathcal{C}(\Delta_i)$ on the polynomial of degree less than r on Δ_i.

2.1 Orthogonal projection operator

Let $\mathcal{P}_h:\mathbb{X}\to\mathbb{X}_h$ be the orthogonal projection operator defined by

$$\langle\mathcal{P}_h\vartheta,v\rangle=\langle\vartheta,v\rangle,\quad v\in\mathbb{X}_h,\quad \vartheta\in\mathbb{X}, \tag{2.14}$$

where $\langle\vartheta,v\rangle=\int_0^1\vartheta(t)v(t)dt$.

We first quote the following Lemma from Chatelin [8].

Lemma 2.1. *Let $\mathcal{P}_h:\mathcal{C}_\Delta\to\mathbb{X}_h$ be the orthogonal projection operator. Then there hold*

i) $\mathcal{P}_h$ is uniformly bounded in infinity norm, i.e., $\exists$ a constant $\hat{p}$ independent of h such that

$$\|\mathcal{P}_h\|_\infty\le\hat{p}<\infty. \tag{2.15}$$

ii) $\|\mathcal{P}_h\vartheta-\vartheta\|_\infty\to0$ as $h\to0$, $\vartheta\in\mathcal{C}_\Delta$.

iii) In particular if $\vartheta\in\mathcal{C}_\Delta^{r+1}$, then

$$\|(\mathcal{I}-\mathcal{P}_h)\vartheta\|_\infty\le ch^{r+1}\|\vartheta^{(r+1)}\|_\infty. \tag{2.16}$$

To apply the modified Galerkin method, we define the operator $\mathcal{T}_h^M:\mathbb{X}\to\mathbb{X}$ (see [10, 14,23]) as

$$\mathcal{T}_h^M(\vartheta):=\mathcal{P}_h\Phi(\mathcal{K}(\vartheta)+f)+\Phi(\mathcal{K}(\mathcal{P}_h\vartheta)+f)-\mathcal{P}_h\Phi(\mathcal{K}(\mathcal{P}_h\vartheta)+f),\quad \vartheta\in\mathbb{X}. \tag{2.17}$$

Then the modified Galerkin method for Eq. (2.13) is seeking an approximate solution $\varrho_h^M\in\mathbb{X}$ such that

$$\varrho_h^M=\mathcal{T}_h^M\varrho_h^M. \tag{2.18}$$

In order to obtain more accurate approximation, we define the iterated modified Galerkin solution by

$$\tilde{\varrho}_h^M=\Phi(\mathcal{K}\varrho_h^M+f). \tag{2.19}$$

Then from (2.10), we can see the corresponding approximations ϑ_h^M and $\tilde{\vartheta}_h^M$ of ϑ are given by

$$\vartheta_h^M=\mathcal{K}(\varrho_h^M)+f,\quad \tilde{\vartheta}_h^M=\mathcal{K}(\tilde{\varrho}_h^M)+f. \tag{2.20}$$

Note that $\varrho_h^M\in\mathbb{X}$. To solve Eq. (2.18), applying $\mathcal{P}_h$ and $(\mathcal{I}-\mathcal{P}_h)$ to the equation, we have

$$\mathcal{P}_h\varrho_h^M=\mathcal{P}_h\Phi(\mathcal{K}(\varrho_h^M)+f). \tag{2.21a}$$

$$(\mathcal{I}-\mathcal{P}_h)\varrho_h^M=(\mathcal{I}-\mathcal{P}_h)\Phi(\mathcal{K}(\mathcal{P}_h\varrho_h^M)+f). \tag{2.21b}$$

Eq. (2.21b) can be written as

$$\varrho_h^M=\mathcal{P}_h\varrho_h^M+(\mathcal{I}-\mathcal{P}_h)\Phi(\mathcal{K}(\mathcal{P}_h\varrho_h^M)+f). \tag{2.22}$$

Substituting (2.22) into (2.21a), we get

$$\mathcal{P}_h\varrho_h^M=\mathcal{P}_h\Phi(\mathcal{K}(\mathcal{P}_h\varrho_h^M+(\mathcal{I}-\mathcal{P}_h)\Phi(\mathcal{K}(\mathcal{P}_h\varrho_h^M)+f))+f). \tag{2.23}$$

We let $\mathcal{W}_h^M=\mathcal{P}_h\varrho_h^M$, then we can seek $\mathcal{W}_h^M\in\mathcal{X}_h$ from the equation

$$\mathcal{W}_h^M=\mathcal{P}_h\Phi(\mathcal{K}(\mathcal{W}_h^M+(\mathcal{I}-\mathcal{P}_h)\Phi(\mathcal{K}(\mathcal{W}_h^M)+(\mathcal{I}-\mathcal{P}_h)f))+f), \tag{2.24}$$

and ϱ_h^M can be obtained using Eq. (2.22) as

$$\varrho_h^M=\mathcal{W}_h^M+(\mathcal{I}-\mathcal{P}_h)\Phi(\mathcal{K}(\mathcal{W}_h^M)+f). \tag{2.25}$$

To solve (2.18), we let

$$\mathcal{F}_h(y)=y-\mathcal{P}_h\Phi(\mathcal{K}(y+(\mathcal{I}-\mathcal{P}_h)\Phi(\mathcal{K}(y)+f))+f). \tag{2.26}$$

The Fréchet derivative of $\mathcal{F}_h$, for any $t\in\mathbb{X}$ is given by

$$\begin{aligned}\mathcal{F}_h'(y)t=&t-\mathcal{P}_h\Phi'(\mathcal{K}(y+(\mathcal{I}-\mathcal{P}_h)\Phi(\mathcal{K}(y)+f))+f)(\mathcal{K}'(y+(\mathcal{I}-\mathcal{P}_h)\Phi(\mathcal{K}(y)+f)))\\&\times(\mathcal{I}-\mathcal{P}_h)\Phi'(\mathcal{K}(y)+f)\mathcal{K})t.\end{aligned} \tag{2.27}$$

Then Eq. (2.25) is equivalent to

$$\mathcal{F}_h(\mathcal{W}_h^M)=0, \tag{2.28}$$

and it is iteratively solved by applying the Newton-Kantorovich method.

3 Superconvergence results

In this section, we analyze the existence and convergence of the approximate and iterated approximate solutions in the modified Galerkin method. To accomplish this, we define $BL(\mathbb{X})$ the space of all bounded linear operators on $\mathbb{X}$.

We first quote the following theorem from [28], which gives us the condition under which the solvability of one equation leads to the solvability of another.

Theorem 3.1 ([28])**.** *Let $\mathcal{X}$ be a Banach space with Ω be an open set and $\widehat{\mathcal{T}}$ and $\widetilde{\mathcal{T}}$ be continuous operators. Let the equation $\vartheta=\widetilde{\mathcal{T}}\vartheta$ has an isolated solution $\tilde{\vartheta}_0\in\Omega$ and let the following conditions be satisfied*

- *(a) The operator $\widehat{\mathcal{T}}$ is Frechet differentiable in some neighborhood of the point $\tilde{\vartheta}_0$, while the linear operator $\mathcal{I}-\widehat{\mathcal{T}'}(\tilde{\vartheta}_o)$ is continuously invertible.*
- *(b) Suppose that for some $\delta>0$ and $0<q<1$, the following inequalities are valid (the number δ is assumed to be so small that the sphere $\|\vartheta-\tilde{\vartheta}_0\|\le\delta$ is contained within Ω)*

$$\sup_{\|\vartheta-\tilde{\vartheta}_0\|\le\delta}\|(\mathcal{I}-\widehat{\mathcal{T}'}(\tilde{\vartheta}_o))^{-1}(\widehat{\mathcal{T}'}(\vartheta)-\widehat{\mathcal{T}'}(\tilde{\vartheta}_o))\|\le q, \tag{3.1a}$$

$$\alpha=\|(\mathcal{I}-\widehat{\mathcal{T}'}(\tilde{\vartheta}_o))^{-1}(\widehat{\mathcal{T}}(\tilde{\vartheta}_o)-\widetilde{\mathcal{T}}(\tilde{\vartheta}_o))\|\le\delta(1-q). \tag{3.1b}$$

Then the equation $\vartheta=\widehat{\mathcal{T}}\vartheta$ *has a unique solution* $\hat{\vartheta}_0$ *in the sphere* $\|\vartheta-\tilde{\vartheta}_0\|\leq\delta$. *Moreover, the inequality*

$$\frac{\alpha}{1+q}\leq\|\hat{\vartheta}_0-\tilde{\vartheta}_0\|\leq\frac{\alpha}{1-q}, \tag{3.2}$$

is valid.

Next we analyze the existence and rates of convergence of the approximative solution ϱ_h^M to ϱ_0. We first give the following lemmas to do this.

Lemma 3.1. *Let* $\varrho_0\in\mathcal{C}^{r_1}[0,1]$ *be the unique solution of Eq.* (2.12). *Then we have the following*

$$\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)\varrho_0\|_\infty=\mathcal{O}(h^{p+p_1}),$$

and

$$\|\mathcal{L}(\mathcal{I}-\mathcal{P}_h)\varrho_0\|_\infty=\mathcal{O}(h^{p+p_2}),$$

where $p=\min\{r_1,r+1\}$, $p_1=\min\{r_1,r+1,\gamma+2\}$, $p_2=\min\{r_1-1,r+1,\gamma+1\}$.

Proof. The proof of the theorem follows from [24]. □

Lemma 3.2. *Let* $\varrho_0\in\mathcal{C}^{r_1}[0,1]$ *be a unique solution of Eq.* (2.12). *Then the following results hold*

$$\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)\|_\infty=\mathcal{O}(h^{p_1}),$$

and

$$\|\mathcal{L}(\mathcal{I}-\mathcal{P}_h)\|_\infty=\mathcal{O}(h^{p_2}),$$

where, $p_1=\min\{r_1,r+1,\gamma+2\}$, $p_2=\min\{r_1-1,r+1,\gamma+1\}$.

Proof. Using orthogonality of $\mathcal{P}_h$, we have

$$\begin{aligned}\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)\varrho_0\|_\infty&=\sup_{t\in[0,1]}|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)\varrho_0(t)|=\sup_{t\in[0,1]}\left|\int_0^1\kappa_t(\chi)(\mathcal{I}-\mathcal{P}_h)\varrho_0(\chi)d\chi\right|\\ &=\sup_{t\in[0,1]}\left|\sum_{i=1}^n\int_{x_{i-1}}^{x_i}(\kappa_t)_i(\chi)(\mathcal{I}-\mathcal{P})(\varrho_0)_i(\chi)d\chi\right|=\sup_{t\in[0,1]}\left|\sum_{i=1}^n\langle(\kappa_t)_i,(\mathcal{I}-\mathcal{P})(\varrho_0)_i\rangle\right|\\ &=\sup_{t\in[0,1]}\left|\sum_{i=1}^n\langle(\mathcal{I}-\mathcal{P})(\kappa_t)_i,(\varrho_0)_i\rangle\right|\leq\sum_{i=1}^n[\|(\mathcal{I}-\mathcal{P})(\kappa_t)_i\|_{2,\Delta_i}\|(\varrho_0)_i\|_{2,\Delta_i}].\end{aligned} \tag{3.3}$$

Now we consider $t\notin\Delta$, i.e., $t\in(x_{i-1},x_i)$, for some $i\in\{1,2,\cdots,n\}$ and $(\kappa_{1t})_j,(\kappa_{2t})_j\in\mathcal{C}^{r_1}(\Delta_j)$, for $j\neq i$, then from Lemma 7.8 of [8], we have for $j\neq i$, and $j=1,2,\dots,n$,

$$\begin{aligned}&\|(\mathcal{I}-\mathcal{P})(\kappa_t)_j\|_{2,\Delta_j}\leq ch_j^p\max(\|(\kappa_{1t})_j^p\|_{2,\Delta_j},\|(\kappa_{2t})_j^p\|_{2,\Delta_j})\\ \leq&ch_j^{p+\frac{1}{2}}\max(\|\kappa_{1t}^{(p)}\|_\infty,\|\kappa_{2t}^{(p)}\|_\infty)=\mathcal{O}(h^{p+\frac{1}{2}}),\end{aligned} \tag{3.4}$$

and on Δ_i,

$$\begin{aligned}
&\|(\mathcal{I}-\mathcal{P})(\kappa_t)_i\|_{2,\Delta_i}\leq ch_i^{p^*}[(\|(\kappa_{1t})_i^{p^*}\|_{2,[t_{i-1},t]})^2+(\|(\kappa_{2t})_i^{p^*}\|_{2,[t,t_i]})^2]^{\frac{1}{2}}\\
\leq & ch_j^{p^*+\frac{1}{2}}[(\|\kappa_{1t}^{(p^*)}\|_\infty)^2+(\|\kappa_{2t}^{(p^*)}\|_\infty)^2]^{\frac{1}{2}}=\mathcal{O}(h^{p^*+\frac{1}{2}}),
\end{aligned}\tag{3.5}$$

where $p^*=\min\{r_1,r+1,\gamma+1\}$, and $p=\min\{r_1,r+1\}$.

Now from estimate (3.3), we have

$$\begin{aligned}
\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)\varrho_0\|_\infty\leq&\sum_{i=1}^n[\|(\mathcal{I}-\mathcal{P})(\kappa_t)_i\|_{2,\Delta_i}\|(\varrho_0)_i\|_{2,\Delta_i}]\\
\leq&\sum_{j=1,j\neq i}^n[\|(\mathcal{I}-\mathcal{P})(\kappa_t)_j\|_{2,\Delta_j}\|(\varrho_0)_j\|_{2,\Delta_j}]+\|(\mathcal{I}-\mathcal{P})(\kappa_t)_i\|_{2,\Delta_i}\|(\varrho_0)_i\|_{2,\Delta_i}\\
\leq& ch^{\frac{1}{2}}\|\varrho_0\|_\infty\Bigg[\sum_{j=1,j\neq i}^n[\|(\mathcal{I}-\mathcal{P})(\kappa_t)_j\|_{2,\Delta_j}]+\|(\mathcal{I}-\mathcal{P})(\kappa_t)_i\|_{2,\Delta_i}\Bigg].
\end{aligned}\tag{3.6}$$

Hence from estimates (3.4)-(3.6), it implies that

$$\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)\|_\infty=\mathcal{O}(h^{\min\{p,p^*+1\}})=\mathcal{O}(h^{\min\{r_1,r+1,\gamma+2\}})=\mathcal{O}(h^{p_1}),\tag{3.7}$$

where $p_1=\min\{r_1,r+1,\gamma+2\}$.

Next, using similar technique of (3.3), we obtain

$$\|\mathcal{L}(\mathcal{I}-\mathcal{P}_h)\varrho_0\|_\infty\leq\sum_{i=1}^n[\|(\mathcal{I}-\mathcal{P})(\ell_t)_i\|_{2,\Delta_i}\|(\varrho_0)_i\|_{2,\Delta_i}].\tag{3.8}$$

Then consider $t\notin\Delta$, i.e., $t\in(x_{i-1},x_i)$, for some $i\in\{1,2,\cdots,n\}$ and $(\ell_{1t})_j,(\ell_{2t})_j\in\mathcal{C}^{r_1-1}(\Delta_j)$, for $j\neq i$, then from Lemma 7.8 of [8], we have for $j\neq i$, and $j=1,2,\ldots,n$,

$$\begin{aligned}
&\|(\mathcal{I}-\mathcal{P})(\ell_t)_j\|_{2,\Delta_j}\leq ch_j^{p^{**}}\max(\|(\ell_{1t})_j^{p^{**}}\|_{2,\Delta_j},\|(\ell_{2t})_j^{p^{**}}\|_{2,\Delta_j})\\
\leq& ch_j^{p^{**}+\frac{1}{2}}\max(\|\ell_{1t}^{(p^{**})}\|_\infty,\|\ell_{2t}^{(p^{**})}\|_\infty)=\mathcal{O}(h^{p^{**}+\frac{1}{2}}),
\end{aligned}\tag{3.9}$$

and on Δ_i,

$$\begin{aligned}
&\|(\mathcal{I}-\mathcal{P})(\ell_t)_i\|_{2,\Delta_i}\leq ch_i^{p^*-1}[(\|(\ell_{1t})_i^{p^*-1}\|_{2,[t_{i-1},t]})^2+(\|(\ell_{2t})_i^{p^*}\|_{2,[t,t_i]})^2]^{\frac{1}{2}}\\
\leq& ch_j^{p^*-\frac{1}{2}}[(\|\ell_{1t}^{(p^*)}\|_\infty)^2+(\|\ell_{2t}^{(p^*)}\|_\infty)^2]^{\frac{1}{2}}=\mathcal{O}(h^{p^*-\frac{1}{2}}),
\end{aligned}\tag{3.10}$$

where $p^*=\min\{r_1,r+1,\gamma+1\}$, and $p^{**}=\min\{r_1-1,r+1\}$.

Now from estimate (3.8) we have

$$\begin{aligned}\|\mathcal{L}(\mathcal{I}-\mathcal{P}_h)\varrho_0\|_\infty \le& \sum_{i=1}^{n}[\|(\mathcal{I}-\mathcal{P})(\ell_t)_i\|_{2,\Delta_i}\|(\varrho_0)_i\|_{2,\Delta_i}]\\ \le& \sum_{j=1,j\neq i}^{n}[\|(\mathcal{I}-\mathcal{P})(\ell_t)_j\|_{2,\Delta_j}\|(\varrho_0)_j\|_{2,\Delta_j}]+\|(\mathcal{I}-\mathcal{P})(\ell_t)_i\|_{2,\Delta_i}\|(\varrho_0)_i\|_{2,\Delta_i}\\ \le& ch^{\frac{1}{2}}\|\varrho_0\|_\infty\left[\sum_{j=1,j\neq i}^{n}[\|(\mathcal{I}-\mathcal{P})(\ell_t)_j\|_{2,\Delta_j}]+\|(\mathcal{I}-\mathcal{P})(\ell_t)_i\|_{2,\Delta_i}\right]. \qquad (3.11)\end{aligned}$$

Hence from estimates (3.9)-(3.11), it implies that

$$\|\mathcal{L}(\mathcal{I}-\mathcal{P}_h)\|_\infty=\mathcal{O}(h^{\min\{p^*,p^{**}\}})=\mathcal{O}(h^{\min\{r_1-1,r+1,\gamma+1\}})=\mathcal{O}(h^{p_2}), \qquad (3.12)$$

where $p_2=\min\{r_1-1,r+1,\gamma+1\}$. Hence the proof follows. □

Lemma 3.3. *Let the operators $\mathcal{T}(\varrho)$ and $\widetilde{\mathcal{T}}_h(\varrho)$ have the Fréchet derivatives $\mathcal{T}'(\varrho)$ and $\widetilde{\mathcal{T}}_h'(\varrho)$, respectively. Then the following hold*

$$\begin{aligned}&\|(\mathcal{I}-\mathcal{P}_h)\widetilde{\mathcal{T}}_h'(\varrho_0)\|_\infty\to 0, \qquad h\to 0,\\ &\|(\mathcal{I}-\mathcal{P}_h)\mathcal{T}'(\varrho_0)\|_\infty\to 0, \qquad h\to 0.\end{aligned}$$

Proof. We have

$$\widetilde{\mathcal{T}}_h'(\varrho_0)=\Phi^{(0,1,0)}(f+\mathcal{K}\mathcal{P}_h\varrho_0)\mathcal{K}\mathcal{P}_h+\Phi^{(0,0,1)}(f+\mathcal{K}\mathcal{P}_h\varrho_0)\mathcal{L}\mathcal{P}_h. \qquad (3.13)$$

With the use of Lemma 3.1, Lipschitz's continuity of $\phi^{(0,1,0)}(\cdot,\vartheta(\cdot),\vartheta'(\cdot))$, $\phi^{(0,0,1)}(\cdot,\vartheta(\cdot),\vartheta'(\cdot))$, and boundedness of $\|\Phi^{(0,1,0)}(f+\mathcal{K}\varrho_0)\|_\infty$ and $\|\Phi^{(0,0,1)}(f+\mathcal{K}\varrho_0)\|_\infty$, we have that

$$\begin{aligned}\|\Phi^{(0,1,0)}(f+\mathcal{K}\mathcal{P}_h\varrho_0)\|_\infty \le& \|\Phi^{(0,1,0)}(f+\mathcal{K}\mathcal{P}_h\varrho_0)-\Phi^{(0,1,0)}(f+\mathcal{K}\varrho_0)\|_\infty+\|\Phi^{(0,1,0)}(f+\mathcal{K}\varrho_0)\|_\infty\\ \le& c_2\{\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)\varrho_0\|_\infty+\|\mathcal{L}(\mathcal{I}-\mathcal{P}_h)\varrho_0\|_\infty\}+\|\Phi^{(0,1,0)}(f+\mathcal{K}\varrho_0)\|_\infty\\ \le& B_1<\infty, \qquad (3.14)\end{aligned}$$

where B_1 is a constant independent of h.

Similarly, we may write that

$$\|\Phi^{(0,0,1)}(f+\mathcal{K}\mathcal{P}_h\varrho_0)\|_\infty\le B_2<\infty, \qquad (3.15)$$

where B_2 is a constant independent of h.

Next, we let $\bar{B}:=\{t\in\mathbb{X}:\|t\|_\infty\le 1\}$ be the closed unit ball in $\mathbb{X}$. Since $\{\mathcal{K}\mathcal{P}_h\}$ and $\{\mathcal{L}\mathcal{P}_h\}$ are sequence of compact operators, using Eq. (3.13), one can show the relatively compactness of the set $S=\{\widetilde{\mathcal{T}}_h'(\varrho_0)\vartheta:\vartheta\in\bar{B},\ n\in N\}$. From Lemma 2.1, it is concluded that

$$\begin{aligned}\|(\mathcal{I}-\mathcal{P}_h)\widetilde{\mathcal{T}}_h'(\varrho_0)\|_\infty=&\sup\{\|(\mathcal{I}-\mathcal{P}_h)\widetilde{\mathcal{T}}_h'(\varrho_0)\vartheta\|_\infty:\vartheta\in\bar{B}\}\\ =&\sup\{\|(\mathcal{I}-\mathcal{P}_h)v\|_\infty:v\in S\}\to 0 \quad \text{as } h\to 0. \qquad (3.16)\end{aligned}$$

Similarly, since $\Phi^{(0,1,0)}(f+\mathcal{K}\varrho_0)$ and $\Phi^{(0,0,1)}(f+\mathcal{K}\varrho_0)$ are bounded and $\mathcal{K}$, $\mathcal{L}$ are compact operators, we can say that

$$\mathcal{T}'(\varrho_0)=\Phi^{(0,1,0)}(f+\mathcal{K}\varrho_0)\mathcal{K}+\Phi^{(0,0,1)}(f+\mathcal{K}\varrho_0)\mathcal{L},$$

is also compact and

$$\|(\mathcal{I}-\mathcal{P}_h)\mathcal{T}'(\varrho_0)\|_\infty\to 0 \quad \text{as} \ \ h\to 0.$$

This complete the proof. □

Theorem 3.2. *Let ϱ_0 is an isolated solution of Eq.* (2.3). *Suppose that $\mathcal{T}'(\varrho_0)$ does not include* 1 *as an eigenvalue. Then there exists a constant $L_1>0$, such that*

$$\|(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}\|_\infty<L_1,$$

for sufficiently small h.

Proof. We consider

$$\begin{aligned}
&\|\mathcal{T}_h^{M'}(\varrho_0)-\mathcal{T}'(\varrho_0)\|_\infty\\
=&\|\mathcal{P}_h\Phi^{(0,1,0)}(f+\mathcal{K}\varrho_0)\mathcal{K}+\mathcal{P}_h\Phi^{(0,0,1)}(f+\mathcal{K}\varrho_0)\mathcal{L}+\Phi^{(0,1,0)}(f+\mathcal{K}(\mathcal{P}_h\varrho_0))\mathcal{K}\mathcal{P}_h\\
&\quad+\Phi^{(0,0,1)}(f+\mathcal{K}(\mathcal{P}_h\varrho_0))\mathcal{L}\mathcal{P}_h-\mathcal{P}_h\Phi^{(0,1,0)}(f+\mathcal{K}(\mathcal{P}_h\varrho_0))\mathcal{K}\mathcal{P}_h\\
&\quad-\mathcal{P}_h\Phi^{(0,0,1)}(f+\mathcal{K}(\mathcal{P}_h\varrho_0))\mathcal{L}\mathcal{P}_h-\Phi^{(0,1,0)}(f+\mathcal{K}\varrho_0)\mathcal{K}-\Phi^{(0,0,1)}(f+\mathcal{K}\varrho_0)\mathcal{L}\|_\infty\\
\le&\|(\mathcal{P}_h-\mathcal{I})[\Phi^{(0,1,0)}(f+\mathcal{K}\varrho_0)\mathcal{K}-\Phi^{(0,1,0)}(f+\mathcal{K}(\varrho_0))\mathcal{K}\mathcal{P}_h]\|_\infty\\
&\quad+\|(\mathcal{P}_h-\mathcal{I})[\Phi^{(0,0,1)}(f+\mathcal{K}\varrho_0)\mathcal{L}-\Phi^{(0,0,1)}(f+\mathcal{K}(\varrho_0))\mathcal{L}\mathcal{P}_h]\|_\infty\\
&\quad+\|(\mathcal{P}_h-\mathcal{I})[\Phi^{(0,1,0)}(f+\mathcal{K}\varrho_0)\mathcal{K}\mathcal{P}_h-\Phi^{(0,1,0)}(f+\mathcal{K}(\mathcal{P}_h\varrho_0))\mathcal{K}\mathcal{P}_h]\|_\infty\\
&\quad+\|(\mathcal{P}_h-\mathcal{I})[\Phi^{(0,0,1)}(f+\mathcal{K}\varrho_0)\mathcal{L}\mathcal{P}_h-\Phi^{(0,0,1)}(f+\mathcal{K}(\mathcal{P}_h\varrho_0))\mathcal{L}\mathcal{P}_h]\|_\infty\\
\le&(\hat{p}+1)\|\Phi^{(0,1,0)}(f+\mathcal{K}\varrho_0)\|_\infty\|\mathcal{K}-\mathcal{K}\mathcal{P}_h\|_\infty+(\hat{p}+1)\|\Phi^{(0,0,1)}(f+\mathcal{K}\varrho_0)\|_\infty\|\mathcal{L}-\mathcal{L}\mathcal{P}_h\|_\infty\\
&\quad+(\hat{p}+1)\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)\varrho_0\|_\infty\|\mathcal{K}\mathcal{P}_h\|_\infty+(\hat{p}+1)\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)\varrho_0\|_\infty\|\mathcal{L}\mathcal{P}_h\|_\infty.
\end{aligned}$$

Using Lemmas 3.1 and 3.2, it follows that

$$\|\mathcal{T}_h^{M'}(\varrho_0)-\mathcal{T}'(\varrho_0)\|_\infty\to 0 \quad \text{as} \ \ h\to 0. \tag{3.17}$$

This implies $\mathcal{T}_h^{M'}(\varrho_0)$ is norm convergent to $\mathcal{T}'(\varrho_0)$. Thus by direct application of Lemma [2], we can conclude that for sufficiently large n,

$$\|(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}\|_\infty<L_1,$$

where $L_1>0$ is a constant. □

Lemma 3.4. *For any $\varrho,\varrho_0\in\mathbb{X}$, the following result hold*

$$\|\mathcal{T}_h^{M'}(\varrho_0)-\mathcal{T}_h^{M'}(\varrho)\|_\infty\le[c\hat{p}MM_1+(1+\hat{p})MM_1\hat{p}^2]\|\varrho_0-\varrho\|_\infty,$$

where c is a constant independent of h.

Proof. For any $\varrho,\varrho_0,y\in\mathbb{X}$, consider

$$\begin{aligned}
&\|[\mathcal{T}_h^{M'}(\varrho_0)-\mathcal{T}_h^{M'}(\varrho)]y\|_\infty\\
=&\|\mathcal{P}_h\Phi^{(0,1,0)}(f+\mathcal{K}\varrho_0)\mathcal{K}y+\mathcal{P}_h\Phi^{(0,0,1)}(f+\mathcal{K}\varrho_0)\mathcal{L}y\\
&+\Phi^{(0,1,0)}(f+\mathcal{K}(\mathcal{P}_h\varrho_0))\mathcal{K}\mathcal{P}_hy+\Phi^{(0,0,1)}(f+\mathcal{K}(\mathcal{P}_h\varrho_0))\mathcal{L}\mathcal{P}_hy\\
&-\mathcal{P}_h\Phi^{(0,1,0)}(f+\mathcal{K}(\mathcal{P}_h\varrho_0))\mathcal{K}\mathcal{P}_hy-\mathcal{P}_h\Phi^{(0,0,1)}(f+\mathcal{K}(\mathcal{P}_h\varrho_0))\mathcal{L}\mathcal{P}_hy\\
&-\mathcal{P}_h\Phi^{(0,1,0)}(f+\mathcal{K}\varrho)\mathcal{K}y-\mathcal{P}_h\Phi^{(0,0,1)}(f+\mathcal{K}\varrho)\mathcal{L}y\\
&-\Phi^{(0,1,0)}(f+\mathcal{K}(\mathcal{P}_h\varrho))\mathcal{K}\mathcal{P}_hy+\Phi^{(0,0,1)}(f+\mathcal{K}(\mathcal{P}_h\varrho))\mathcal{L}\mathcal{P}_hy\\
&+\mathcal{P}_h\Phi^{(0,1,0)}(f+\mathcal{K}(\mathcal{P}_h\varrho))\mathcal{K}\mathcal{P}_hy+\mathcal{P}_h\Phi^{(0,0,1)}(f+\mathcal{K}(\mathcal{P}_h\varrho))\mathcal{L}\mathcal{P}_h\|_\infty\\
\le&\|\mathcal{P}_h[\Phi^{(0,1,0)}(f+\mathcal{K}\varrho_0)-\Phi^{(0,1,0)}(f+\mathcal{K}\varrho)]\mathcal{K}y\|_\infty\\
&+\|\mathcal{P}_h[\Phi^{(0,0,1)}(f+\mathcal{K}\varrho_0)-\Phi^{(0,0,1)}(f+\mathcal{K}\varrho)]\mathcal{L}y\|_\infty\\
&+\|(\mathcal{I}-\mathcal{P}_h)[\Phi^{(0,1,0)}(f+\mathcal{K}\mathcal{P}_h\varrho_0)-\Phi^{(0,1,0)}(f+\mathcal{K}\mathcal{P}_h\varrho)]\mathcal{K}\mathcal{P}_hy\|_\infty\\
&+\|(\mathcal{I}-\mathcal{P}_h)[\Phi^{(0,0,1)}(f+\mathcal{K}\mathcal{P}_h\varrho_0)-\Phi^{(0,0,1)}(f+\mathcal{K}\mathcal{P}_h\varrho)]\mathcal{L}\mathcal{P}_hy\|_\infty\\
\le&\hat{p}\|[\Phi^{(0,1,0)}(f+\mathcal{K}\varrho_0)-\Phi^{(0,1,0)}(f+\mathcal{K}\varrho)]\mathcal{K}y\|_\infty\\
&+\hat{p}\|[\Phi^{(0,0,1)}(f+\mathcal{K}\varrho_0)-\Phi^{(0,0,1)}(f+\mathcal{K}\varrho)]\mathcal{L}y\|_\infty\\
&+(1+\hat{p})\|[\Phi^{(0,1,0)}(f+\mathcal{K}\mathcal{P}_h\varrho_0)-\Phi^{(0,1,0)}(f+\mathcal{K}\mathcal{P}_h\varrho)]\mathcal{K}\mathcal{P}_hy\|_\infty\\
&+(1+\hat{p})\|[\Phi^{(0,0,1)}(f+\mathcal{K}\mathcal{P}_h\varrho_0)-\Phi^{(0,0,1)}(f+\mathcal{K}\mathcal{P}_h\varrho)]\mathcal{L}\mathcal{P}_hy\|_\infty\\
\le&\hat{p}\|\mathcal{K}(\varrho_0-\varrho)\|_\infty[\|\mathcal{K}y\|_\infty+\|\mathcal{L}y\|_\infty]\\
&+(1+\hat{p})\|\mathcal{K}\mathcal{P}_h(\varrho_0-\varrho)\|_\infty[\|\mathcal{K}\mathcal{P}_hy\|_\infty+\|\mathcal{L}\mathcal{P}_hy\|_\infty].
\end{aligned} \tag{3.18}$$

Now using the estimate (2.7), we have

$$\begin{aligned}
\|[\mathcal{T}_h^{M'}(\varrho_0)-\mathcal{T}_h^{M'}(\varrho)]y\|_\infty\le&c\hat{p}MM_1\|\varrho-\varrho_0\|_\infty\|y\|_\infty+(1+\hat{p})MM_1\hat{p}^2\|\varrho-\varrho_0\|_\infty\|y\|_\infty\\
=&[c\hat{p}MM_1+(1+\hat{p})MM_1\hat{p}^2]\|\varrho-\varrho_0\|_\infty\|y\|_\infty.
\end{aligned}$$

This implies

$$\|\mathcal{T}_h^{M'}(\varrho_0)-\mathcal{T}_h^{M'}(\varrho)\|_\infty\le[c\hat{p}MM_1+(1+\hat{p})MM_1\hat{p}^2]\|\varrho-\varrho_0\|_\infty.$$

Hence the proof follows. $\square$

Theorem 3.3. *Let ϱ_0 be an isolated solution of Eq.* (2.3)*. Assume that 1 is not an eigen value of $\mathcal{T}'(\varrho_0)$. Then Eq.* (2.18) *has a unique solution $\varrho_h^M \in B(\varrho_0,\delta)=\{\varrho : \|\varrho-\varrho_0\|_\infty<\delta\}$, then there exists a constant $0<q<1$, independent of h such that*

$$\frac{\alpha_h}{1+q} \le \|\varrho_h^M-\varrho_0\|_\infty \le \frac{\alpha_h}{1-q},$$

where

$$\alpha_h=\|(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}(\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0))\|_\infty.$$

Proof. From Theorem 3.2, we have that there exists a constant L_1 such that for some sufficiently large n,

$$\|(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}\|_\infty \le L_1<\infty.$$

Following the Lemma 3.4, for any $\varrho\in B(\varrho_0,\delta)$, we have

$$\|\mathcal{T}_h^{M'}(\varrho_0)-\mathcal{T}_h^{M'}(\varrho)\|_\infty \le [c\hat{p}MM_1+(1+\hat{p})MM_1\hat{p}^2]\|\varrho_0-\varrho\|_\infty.$$

Thus

$$\begin{aligned}&\sup_{\|\varrho-\varrho_0\|\le\delta}\|(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}(\mathcal{T}_h^{M'}(\varrho_0)-\mathcal{T}_h^{M'}(\varrho))\|_\infty\\ \le& L_1[c\hat{p}MM_1+(1+\hat{p})MM_1\hat{p}^2]\delta\le q,\end{aligned}$$

where, δ is chosen so that $0<q<1$. This proves Eq. (3.1a) of Theorem 3.1.

With the use of Lemma 3.1, Theorem 3.2, and Lipschitz continuity of Φ, we have

$$\begin{aligned}\alpha_h=&\|(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}(\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0))\|_\infty\\ \le& L_1\|\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)\|_\infty\\ =&L_1\|\mathcal{P}_h\Phi(\mathcal{K}\varrho_0+f)+\Phi(\mathcal{K}\mathcal{P}_h\varrho_0+f)-\mathcal{P}_h\Phi(\mathcal{K}\mathcal{P}_h\varrho_0+f)-\Phi(\mathcal{K}\varrho_0+f)\|_\infty\\ =&L_1\|(\mathcal{P}_h-\mathcal{I})[\Phi(\mathcal{K}\varrho_0+f)-\Phi(\mathcal{K}\mathcal{P}_h\varrho_0+f)]\|_\infty\\ \le& L_1c(1+\hat{p})[\|\mathcal{K}(\mathcal{P}_h-\mathcal{I})\varrho_0\|_\infty+\|\mathcal{L}(\mathcal{P}_h-\mathcal{I})\varrho_0\|_\infty]\to 0 \quad \text{as } h\to 0.\end{aligned} \tag{3.19}$$

By choosing h sufficiently small so that $\alpha_h\le\delta(1-q)$, Eq. (3.2) of Theorem 3.1 is satisfied. Consequently, by using Theorem 3.1, we find

$$\frac{\alpha_h}{1+q} \le \|\varrho_h^M-\varrho_0\|_\infty \le \frac{\alpha_h}{1-q},$$

where

$$\alpha_h=\|(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}(\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0))\|_\infty.$$

This completes the proof. □

Theorem 3.4. *Let ϱ_0 be an isolated solution of Eq. (2.3) and ϱ_h^M be the modified Galerkin approximation of ϱ_0. Then the followings convergence rates hold*

$$\begin{aligned}&\|\varrho_h^M-\varrho_0\|_\infty=\mathcal{O}(h^{p+p_2}),\\&\|\vartheta_h^M-\vartheta_0\|_\infty=\mathcal{O}(h^{p+p_2}),\end{aligned}$$

where $p=\min\{r_1,r+1\}$ and $p_2=\min\{r_1-1,r+1,\gamma+1\}$.

Proof. From Theorem 3.3, we have

$$\frac{\alpha_h}{1+q}\le\|\varrho_h^M-\varrho_0\|_\infty\le\frac{\alpha_h}{1-q},$$

where

$$\alpha_h=\|(\mathcal{I}-{\mathcal{T}_h}^{M'}(\varrho_0))^{-1}({\mathcal{T}_h}^{M}(\varrho_0)-\mathcal{T}(\varrho_0))\|_\infty.$$

Using Lipschitz continuity of Φ, and the results of Lemma 3.1 and Theorem 3.2, we have

$$\begin{aligned}\|\varrho_h^M-\varrho_0\|_\infty&\le\frac{\alpha_h}{1-q}\le\frac{1}{1-q}\|(\mathcal{I}-{\mathcal{T}_h}^{M'}(\varrho_0))^{-1}({\mathcal{T}_h}^{M}(\varrho_0)-\mathcal{T}(\varrho_0))\|_\infty\\&\le c\|(\mathcal{I}-{\mathcal{T}_h}^{M'}(\varrho_0))^{-1}\|_\infty\|{\mathcal{T}_h}^{M}(\varrho_0)-\mathcal{T}(\varrho_0)\|_\infty\\&\le cL_1(1+\hat{p})[\|\mathcal{K}(\mathcal{P}_h-\mathcal{I})\varrho_0\|_\infty+\|\mathcal{L}(\mathcal{P}_h-\mathcal{I})\varrho_0\|_\infty]\\&=\mathcal{O}(h^{p+p_2}).\end{aligned}\tag{3.20}$$

Now from estimate (3.20), we obtain

$$\begin{aligned}\|\vartheta_h^M-\vartheta_0\|_\infty&=\|\mathcal{K}\varrho_h^M-\mathcal{K}\varrho_0\|_\infty=\|\mathcal{K}(\varrho_h^M-\varrho_0)\|_\infty\\&\le M_1\|\varrho_h^M-\varrho_0\|_\infty=\mathcal{O}(h^{p+p_2}).\end{aligned}\tag{3.21}$$

Hence the proof follows. □

Next we analyze the superconvergence results for iterated modified Galerkin approximation. To accomplish this, we must first give the following Lemma.

Lemma 3.5. *Let $\tilde{\varrho}_h^M$ be the iterated modified Galerkin approximation of ϱ_0. Then there hold*

$$\begin{aligned}\|\tilde{\varrho}_h^M-\varrho_0\|_\infty\le& cM_1M_2\|\varrho_h^M-\varrho_0\|_\infty^2+\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty\\&+\|\mathcal{L}(\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty.\end{aligned}$$

Proof. Recall that from Theorem 3.2, we find

$$\|(\mathcal{I}-(\mathcal{T}_h^M)'(\varrho_0))^{-1}\|_\infty\le L_1<\infty.\tag{3.22}$$

Consider

$$\begin{aligned}&\varrho_h^M-\varrho_0=\mathcal{T}_h^M(\varrho_h^M)-\mathcal{T}(\varrho_0)\\&=\mathcal{T}_h^M(\varrho_h^M)-\mathcal{T}_h^M(\varrho_0)-\mathcal{T}_h^{M'}(\varrho_0)(\varrho_h^M-\varrho_0)+\mathcal{T}_h^{M'}(\varrho_0)(\varrho_h^M-\varrho_0)+\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0).\end{aligned}\tag{3.23}$$

This implies

$$
\begin{aligned}
&(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))(\varrho_h^M-\varrho_0)\\
=&\mathcal{T}_h^M(\varrho_h^M)-\mathcal{T}_h^M(\varrho_0)-\mathcal{T}_h^{M'}(\varrho_0)(\varrho_h^M-\varrho_0)+\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0). \qquad (3.24)
\end{aligned}
$$

Using mean-value theorem, we obtain

$$
\begin{aligned}
\varrho_h^M-\varrho_0=&(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}[\mathcal{T}_h^M(\varrho_h^M)-\mathcal{T}_h^M(\varrho_0)-\mathcal{T}_h^{M'}(\varrho_0)(\varrho_h^M-\varrho_0)+\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\\
=&(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}[\mathcal{T}_h^M(\varrho_h^M)-\mathcal{T}_h^M(\varrho_0)-\mathcal{T}_h^{M'}(\varrho_0)(\varrho_h^M-\varrho_0)]\\
&+(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\\
=&(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}[\mathcal{T}_h^{M'}(\varrho_0+\theta_1(\varrho_h^M-\varrho_0))-\mathcal{T}_h^{M'}(\varrho_0)](\varrho_h^M-\varrho_0)\\
&+(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)], \qquad (3.25)
\end{aligned}
$$

where $0<\theta_1<1$.

Operating $\mathcal{K}$ on both sides of the above equation

$$
\begin{aligned}
\|\mathcal{K}(\varrho_h^M-\varrho_0)\|_\infty=&\|\mathcal{K}(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}\|_\infty\|[\mathcal{T}_h^{M'}(\varrho_0+\theta_1(\varrho_h^M-\varrho_0))-\mathcal{T}_h^{M'}(\varrho_0)](\varrho_h^M-\varrho_0)\|_\infty\\
&+\|\mathcal{K}(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty. \qquad (3.26)
\end{aligned}
$$

Now

$$
\begin{aligned}
&\|\mathcal{K}(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}y\|_\infty\le M_1\|(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}y\|_\infty\\
\le& M_1\|(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}\|_\infty\|y\|_\infty\le M_1L\|y\|_\infty. \qquad (3.27)
\end{aligned}
$$

From Eq. (3.26), we have

$$
\begin{aligned}
\|\mathcal{K}(\varrho_h^M-\varrho_0)\|_\infty=&M_1L\|\mathcal{T}_h^{M'}(\varrho_0+\theta_1(\varrho_h^M-\varrho_0))-\mathcal{T}_h^{M'}(\varrho_0)\|_\infty\|\varrho_h^M-\varrho_0\|_\infty\\
&+\|\mathcal{K}(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty. \qquad (3.28)
\end{aligned}
$$

We have

$$
(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}=\mathcal{I}+(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}\mathcal{T}_h^{M'}(\varrho_0). \qquad (3.29)
$$

Using the above identity (3.29) in the second part of the equation (3.28), we obtain

$$
\begin{aligned}
&\|\mathcal{K}(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty\\
=&\|\mathcal{K}\{I+(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}\mathcal{T}_h^{M'}(\varrho_0)\}[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty\\
\le&\|\mathcal{K}[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty+\|\mathcal{K}(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}\mathcal{T}_h^{M'}(\varrho_0)[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty\\
\le&\|\mathcal{K}[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty+M_1L\|\mathcal{T}_h^{M'}(\varrho_0)[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty. \qquad (3.30)
\end{aligned}
$$

Note that

$$\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)=(\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)],$$

and

$$\begin{aligned}&\mathcal{T}_h^{M'}(\varrho_0)[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\\=&\mathcal{P}_h[\Phi^{(0,1,0)}(\mathcal{K}\varrho_0+f)\mathcal{K}+\Phi^{(0,0,1)}(\mathcal{K}\varrho_0+f)\mathcal{L}]\times[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)].\end{aligned}\tag{3.31}$$

Hence from estimates (3.28)-(3.31), we have

$$\begin{aligned}&\|\mathcal{K}(\varrho_h^M-\varrho_0)\|_\infty\\=&M_1L\|\mathcal{T}_h^{M'}(\varrho_0+\theta_1(\varrho_h^M-\varrho_0))-\mathcal{T}_h^{M'}(\varrho_0)\|_\infty\|\varrho_h^M-\varrho_0\|_\infty\\&+\|\mathcal{K}[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty+M_1\|\mathcal{T}_h^{M'}(\varrho_0)[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty\\\le&cMM_1\|\varrho_h^M-\varrho_0\|_\infty^2+\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty\\&+M_1\|\mathcal{P}_h[\Phi^{(0,1,0)}(\mathcal{K}\varrho_0+f)\mathcal{K}+\Phi^{(0,0,1)}(\mathcal{K}\varrho_0+f)\mathcal{L}](\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty\\\le&cMM_1\|\varrho_h^M-\varrho_0\|_\infty^2+\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty\\&+M_1\hat{p}\|[\Phi^{(0,1,0)}(\mathcal{K}\varrho_0+f)\|_\infty\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty\\&+M_1\hat{p}\|[\Phi^{(0,0,1)}(\mathcal{K}\varrho_0+f)\|_\infty\|\mathcal{L}(\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty.\end{aligned}\tag{3.32}$$

Operating $\mathcal{L}$ on both sides of the equation (3.25), we obtain

$$\begin{aligned}\|\mathcal{L}(\varrho_h^M-\varrho_0)\|_\infty=&\|\mathcal{L}(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}\|_\infty\|[\mathcal{T}_h^{M'}(\varrho_0+\theta_1(\varrho_h^M-\varrho_0))-\mathcal{T}_h^{M'}(\varrho_0)](\varrho_h^M-\varrho_0)\|_\infty\\&+\|\mathcal{L}(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty.\end{aligned}\tag{3.33}$$

Now

$$\begin{aligned}&\|\mathcal{L}(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}y\|_\infty\le M_2\|(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}y\|_\infty\\\le&M_2\|(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}\|_\infty\|y\|_\infty\le M_2L\|y\|_\infty.\end{aligned}\tag{3.34}$$

Then from equations (3.33) and (3.34), we have

$$\begin{aligned}\|\mathcal{L}(\varrho_h^M-\varrho_0)\|_\infty=&M_2L\|\mathcal{T}_h^{M'}(\varrho_0+\theta_1(\varrho_h^M-\varrho_0))-\mathcal{T}_h^{M'}(\varrho_0)\|_\infty\|\varrho_h^M-\varrho_0\|_\infty\\&+\|\mathcal{L}(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty.\end{aligned}\tag{3.35}$$

Using the identity (3.29) in the second part of the equation (3.35), we obtain

$$\begin{aligned}&\|\mathcal{L}(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty\\=&\|\mathcal{L}\{I+(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}\mathcal{T}_h^{M'}(\varrho_0)\}[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty\\\le&\|\mathcal{L}[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty+\|\mathcal{L}(\mathcal{I}-\mathcal{T}_h^{M'}(\varrho_0))^{-1}\mathcal{T}_h^{M'}(\varrho_0)[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty\\\le&\|\mathcal{L}[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty+M_2L\|\mathcal{T}_h^{M'}(\varrho_0)[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty.\end{aligned}\tag{3.36}$$

Note that

$$\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)=(\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)], \tag{3.37}$$

and

$$\begin{aligned}&\mathcal{T}_h^{M'}(\varrho_0)[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\\ =&\mathcal{P}_h[\Phi^{(0,1,0)}(\mathcal{K}\varrho_0+f)\mathcal{K}+\Phi^{(0,0,1)}(\mathcal{K}\varrho_0+f)\mathcal{L}][\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)].\end{aligned} \tag{3.38}$$

Combining estimates (3.33)-(3.38), we have

$$\begin{aligned}&\|\mathcal{L}(\varrho_h^M-\varrho_0)\|_\infty\\ \le&M_2L\|\mathcal{T}_h^{M'}(\varrho_0+\theta_1(\varrho_h^M-\varrho_0))-\mathcal{T}_h^{M'}(\varrho_0)\|_\infty\|\varrho_h^M-\varrho_0\|_\infty\\ &+\|\mathcal{L}[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty+M_1\|\mathcal{T}_h^{M'}(\varrho_0)[\mathcal{T}_h^M(\varrho_0)-\mathcal{T}(\varrho_0)]\|_\infty\\ \le&cMM_2\|\varrho_h^M-\varrho_0\|_\infty^2+\|\mathcal{L}(\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty\\ &+M_1\|\mathcal{P}_h[\Phi^{(0,1,0)}(\mathcal{K}\varrho_0+f)\mathcal{K}+\Phi^{(0,0,1)}(\mathcal{K}\varrho_0+f)\mathcal{L}](\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty\\ \le&cMM_2\|\varrho_h^M-\varrho_0\|_\infty^2+\|\mathcal{L}(\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty\\ &+M_2\hat{p}\|[\Phi^{(0,1,0)}(\mathcal{K}\varrho_0+f)\|_\infty\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty\\ &+M_2\hat{p}\|[\Phi^{(0,0,1)}(\mathcal{K}\varrho_0+f)\|_\infty\|\mathcal{L}(\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty.\end{aligned} \tag{3.39}$$

Then from (3.32) and (3.39), it follows that

$$\begin{aligned}&\|\tilde{\varrho}_h^M-\varrho_0\|_\infty\\ \le&cMM_2\|\varrho_h^M-\varrho_0\|_\infty^2+\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty\\ &+\|\mathcal{L}(\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty.\end{aligned} \tag{3.40}$$

Hence the proof follows. □

Theorem 3.5. *Let $\tilde{\varrho}_h^M$ be the iterated modified Galerkin approximation of ϱ_0. Then the following hold*

$$\begin{aligned}\|\tilde{\varrho}_h^M-\varrho_0\|_\infty&=\mathcal{O}(h^{p+2p_2}),\\ \|\tilde{\vartheta}_h^M-\vartheta_0\|_{L^2}&=\mathcal{O}(h^{p+2p_2}),\end{aligned}$$

where $p=\min\{r_1,r+1\}$ and $p_2=\min\{r_1-1,r+1,\gamma+1\}$.

Proof. From the results of Lemma 3.5, we have

$$\begin{aligned}\|\tilde{\varrho}_h^M-\varrho_0\|_\infty\le&cMM_2\|\varrho_h^M-\varrho_0\|_\infty^2+\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty\\ &+\|\mathcal{L}(\mathcal{I}-\mathcal{P}_h)[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty\\ \le&cMM_2\|\varrho_h^M-\varrho_0\|_\infty^2+\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)\|_\infty\|[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty\\ &+\|\mathcal{L}(\mathcal{I}-\mathcal{P}_h)\|_\infty\|[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty.\end{aligned} \tag{3.41}$$

Now using the Lipschitz continuity of $\Phi(\cdot)$, we have

$$\|[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty=\|\mathcal{K}(\mathcal{I}-\mathcal{P}_h)\varrho_0\|_\infty+\|\mathcal{L}(\mathcal{I}-\mathcal{P}_h)\varrho_0\|_\infty. \tag{3.42}$$

Then from the result of Lemma 3.1 and estimate (3.42), we get

$$\|[\Phi(\mathcal{K}(\mathcal{P}_h\varrho_0)+f)-\Phi(\mathcal{K}\varrho_0+f)]\|_\infty=\mathcal{O}(h^{p+p_1})+\mathcal{O}(h^{p+p_2}). \tag{3.43}$$

Again by Lemma 3.2, Theorem 3.4 and estimates (3.42), (3.43), we obtain

$$\|\tilde{\varrho}_h^M-\varrho_0\|_\infty=\mathcal{O}(h^{p+2p_2}). \tag{3.44}$$

Finally from the estimate (2.20) and the above result, it follows that

$$\begin{aligned}\|\tilde{\vartheta}_h^M-\vartheta_0\|_\infty&=\|\mathcal{K}\tilde{\varrho}_h^M-\mathcal{K}\varrho_0\|_\infty=\|\mathcal{K}(\tilde{\varrho}_h^M-\varrho_0)\|_\infty\\&\le M_1\|\tilde{\varrho}_h^M-\varrho_0\|_\infty=\mathcal{O}(h^{p+2p_2}).\end{aligned}$$

This completes the proof. □

4 Numerical example

In this section, three numerical examples are given to illustrate the convergence results. Choosing the approximating subspaces $\mathbb{X}_n$ to be the space of piecewise linear $(r=1)$ functions, we give the errors in infinity norm in the following Tables. For computations we use the Newton-Kantorovich method to generate the numerical algorithms, which are compiled by using Matlab. In Tables 1, 3, and 5, we present the errors in Galerkin and iterated Galerkin methods with approximating subspace as the space of piecewise linear functions, and in Tables 2, 4, and 6, we have given the errors for Galerkin and iterated Galerkin methods with approximating subspace as the space of piecewise linear functions. We denote the Galerkin, iterated Galerkin, modified Galerkin and iterated modified Galerkin solutions by ϑ_n, $\tilde{\vartheta}_n$, ϑ_n^M, and $\tilde{\vartheta}_n^M$ respectively in the following tables.

Note that, for $r=1$, the expected orders of convergence for Galerkin, iterated Galerkin, modified Galerkin, and iterated modified Galerkin methods are $a=2$, $b=3$, $c=3$ and $d=4$, respectively.

Example 4.1. We consider the following two point boundary value problem

$$\begin{aligned}&(\vartheta'(t))'=-(2te^{\vartheta}\vartheta'+2e^{\vartheta}),\\&\vartheta(0)=\ln\Big(\frac{1}{4}\Big),\quad \vartheta(1)=\ln\Big(\frac{1}{5}\Big).\end{aligned}$$

Then the transformed integral equation as follows

$$\vartheta(t)=f(t)+\int_0^1\kappa(t,\chi)\phi(\chi,\vartheta(\chi),\vartheta'(\chi))d\chi,\quad 0\le x\le 1,$$

Table 1: Galerkin and iterated Galerkin methods.

n	$\|\vartheta-\vartheta_n\|_\infty$	a	$\|\vartheta-\tilde{\vartheta}_n\|_\infty$	b
2	0.107×10^{-1}	-	0.122×10^{-3}	-
4	0.262×10^{-2}	2.03	0.158×10^{-4}	2.94
8	0.650×10^{-3}	2.01	0.181×10^{-5}	3.13
16	0.162×10^{-3}	2.01	0.219×10^{-6}	3.04
32	0.406×10^{-4}	2.00	0.293×10^{-7}	2.91
64	0.101×10^{-4}	2.00	0.376×10^{-8}	2.96

Table 2: Modified Galerkin and iterated modified Galerkin methods.

n	$\|\vartheta-\vartheta_n^M\|_\infty$	c	$\|\vartheta-\tilde{\vartheta}_n^M\|_\infty$	d
2	1.32×10^{-4}	-	3.11×10^{-4}	-
4	1.59×10^{-5}	3.05	1.92×10^{-5}	4.01
8	1.81×10^{-6}	3.13	1.22×10^{-6}	3.97
16	2.21×10^{-7}	3.03	7.18×10^{-8}	4.08
32	2.89×10^{-8}	2.93	4.47×10^{-9}	4.00
64	4.10×10^{-9}	2.81	2.69×10^{-10}	4.05

with $f(t)=\ln(\frac{1}{4})+\ln(\frac{4}{5})t$, $\phi(\chi,\vartheta(\chi),\vartheta'(\chi))=-(2\chi e^{\vartheta}\vartheta'+2e^{\vartheta})$, $\vartheta(t)=\ln(\frac{1}{4+t^2})$, and

$$\kappa(t,\chi)=\begin{cases}-t(1-\chi), & 0\le t\le\chi,\\ -\chi(1-t), & \chi\le t\le 1.\end{cases}$$

Example 4.2. Consider the following two point boundary value problem

$$\begin{aligned}&(\vartheta'(t))'=-\vartheta' e^{\vartheta},\\ &\vartheta(0)=\ln\Big(\frac{1}{2}\Big),\quad \vartheta(1)=\ln\Big(\frac{1}{3}\Big).\end{aligned}$$

Then the transformed integral equation as follows

$$\vartheta(t)=f(t)+\int_0^1\kappa(t,\chi)\phi(\chi,\vartheta(\chi),\vartheta'(\chi))d\chi,\quad 0\le t\le 1,$$

where $f(t)=\ln(\frac{1}{2})+\ln(\frac{2}{3})t$, $\phi(\chi,\vartheta(\chi),\vartheta'(\chi))=-e^{\vartheta}\vartheta'$, $\vartheta(t)=\ln(\frac{1}{2+t})$, and

$$\kappa(t,\chi)=\begin{cases}-t(1-\chi), & 0\le t\le\chi,\\ -\chi(1-t), & \chi\le t\le 1.\end{cases}$$

Example 4.3. Consider the following two point boundary value problem

$$\begin{aligned}&(t^{\alpha}\vartheta'(t))'=t^{\alpha+\beta-2}(\beta t\vartheta'(t)+\beta(\alpha+\beta-1)\vartheta(t)),\quad t\in[0,1],\\ &\vartheta(0)=1,\quad \vartheta(1)=e.\end{aligned}$$

Table 3: Galerkin and iterated Galerkin methods.

n	$\|\vartheta-\vartheta_n\|_\infty$	a	$\|\vartheta-\tilde{\vartheta}_n\|_\infty$	b
2	0.451×10^{-2}	-	0.105×10^{-3}	-
4	0.121×10^{-2}	1.90	0.144×10^{-4}	2.87
8	0.311×10^{-3}	1.96	0.187×10^{-5}	2.94
16	0.784×10^{-4}	1.99	0.241×10^{-6}	2.96
32	0.196×10^{-4}	2.00	0.315×10^{-7}	2.94
64	0.486×10^{-5}	2.01	0.452×10^{-8}	2.80

Table 4: Modified Galerkin and iterated modified Galerkin methods.

n	$\|\vartheta-\vartheta_n^M\|_\infty$	c	$\|\vartheta-\tilde{\vartheta}_n^M\|_\infty$	d
2	1.13×10^{-4}	-	1.42×10^{-4}	-
4	1.50×10^{-5}	2.92	1.05×10^{-5}	3.75
8	1.92×10^{-6}	2.96	7.7×10^{-7}	3.76
16	2.42×10^{-7}	2.98	5.42×10^{-8}	3.82
32	3.01×10^{-8}	3.00	3.52×10^{-9}	3.94
64	4.10×10^{-9}	2.87	2.24×10^{-10}	3.97

Table 5: Galerkin and iterated Galerkin methods.

n	$\|\vartheta-\vartheta_n\|_\infty$	a	$\|\vartheta-\tilde{\vartheta}_n\|_\infty$	b
2	0.588×10^{-1}	-	0.772×10^{-2}	-
4	0.155×10^{-1}	1.92	0.108×10^{-2}	2.84
8	0.388×10^{-2}	1.99	0.130×10^{-3}	3.05
16	0.967×10^{-3}	2.00	0.158×10^{-4}	3.03
32	0.242×10^{-3}	2.00	0.228×10^{-5}	2.80
64	0.603×10^{-4}	2.01	0.314×10^{-6}	2.85

For $\alpha=0$, the transformed integral equation as follows

$$\vartheta(t)=f(t)+\int_0^1 \kappa(t,\chi)\phi(\chi,\vartheta(\chi),\vartheta'(\chi))d\chi, \quad 0\le t\le 1,$$

where $f(t)=1+te-t$, $\vartheta(t)=e^{t^\beta}$, and

$$\kappa(t,\chi)=\begin{cases} -t(1-\chi), & 0\le t\le\chi, \\ -\chi(1-t), & \chi\le t\le 1. \end{cases}$$

We have calculated the following table for $\beta=2$.

From Tables 2, 4 and 6, we can observe that the approximate solution in the iterated modified Galerkin technique has higher convergence rates than the approximate solution in modified Galerkin technique. Also, comparing these tables with the results of Tables 1, 3 and 5, we also see that the iterated M-Galerkin method gives better convergence rates than classical Galerkin and iterated Galerkin method.

Table 6: Modified Galerkin and iterated modified Galerkin methods.

n	$\|\vartheta-\vartheta_n^M\|_\infty$	c	$\|\vartheta-\tilde{\vartheta}_n^M\|_\infty$	d
2	0.825×10^{-2}	-	0.166×10^{-1}	-
4	0.109×10^{-2}	2.92	0.110×10^{-2}	3.92
8	0.130×10^{-3}	3.06	6.55×10^{-5}	4.06
16	0.160×10^{-4}	3.02	4.06×10^{-6}	4.01
32	0.203×10^{-5}	2.97	2.62×10^{-7}	3.96
64	0.241×10^{-6}	3.07	1.72×10^{-8}	3.92

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