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A Fourier Matching Method for Analyzing Resonances in a Sound-Hard Slab with Subwavelength Holes

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Abstract. This paper presents a Fourier matching method to rigorously study resonances in a sound-hard slab with a finite number of narrow cylindrical holes. The cross sections of the holes, of diameters $\mathcal{O}(h)$ for $h \ll 1$, can be arbitrarily shaped. Outside the slab, a sound field can be represented in terms of its normal derivatives on the apertures of the holes. Inside each hole, the field can be represented in terms of a countable Fourier basis due to the zero Neumann boundary condition on the side surface. The countably infinite Fourier coefficients for all the holes constitute the unknowns. Matching the two field representatives leads to a countable-dimensional linear system governing the unknowns. Due to the invertibility of a principal submatrix of the infinite-dimensional coefficient matrix, we reduce the linear system to a finite-dimensional one. Resonances are those when the finite-dimensional linear system becomes singular. We derive asymptotic formulae for the resonances in the subwavelength structure for $h \ll 1$. They reveal that a sound field with its real frequency close to a resonance frequency can be enhanced by a magnitude $\mathcal{O}(h^{-2})$. Numerical experiments are carried out to validate the proposed resonance formulae.

AMS subject classifications: 35B34, 35B40, 35J05, 35P20

Key words: Acoustic scattering problem, resonance frequency, subwavelength structure, Helmholtz equation, field enhancement.

1 Introduction

Subwavelength structures have attracted great attentions in the area of wave scattering problems in the past decades [2, 9, 10, 12, 15, 16, 29, 32, 35–37]. People have experimentally observed and also numerically verified that subwavelength structures could own some exclusive features, such as extraordinary optical transmission and local field enhancement, showing great potentials in areas such as biological sensing and imaging,

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microscopy, spectroscopy and communication [23, 31]. It has now been well-known that these features are mostly caused by the existence of resonances of high quality factors in subwavelength structures.

Let a scattering problem be defined in a given subwavelength structure. Mathematically, a resonance refers to a complex frequency k, at which the scattering problem loses uniqueness. The quality factor of this resonance is defined by

$$Q = \frac{\mathrm{Re}(k)}{2\mathrm{Im}(k)},$$

which measures how great a wave field of frequency $\mathrm{Re}(k)$ can be enhanced in the structure. Consequently, it is highly desired to design a subwavelength structure with resonances close enough to the real axis.

To this purpose, the existing literature has made great efforts in the past to quantitatively analyze resonances in subwavelength structures by either developing effective computational methods or proposing rigorous mathematical theories [3, 6–8, 11, 14, 17–21, 25–28, 33, 34]. Among the existing theories, roughly two types of methods have been proposed: boundary-integral-equation (BIE) methods and matched-asymptotics (MA) methods, mainly for two-dimensional (2D) subwavelength structures. Bonnetier and Triki [7] developed a BIE method to study resonances in a perfectly conducting half plane with a subwavelength cavity and firstly obtained asymptotic formulae for the resonances. Subsequently, Babadjian *et al.* [3] used this method to study resonances by two interacting subwavelength cavities, Lin and Zhang developed a simplified BIE method to study resonances in a slab with a single slit [26], periodic slits [27, 28], or a periodic array of two subwavelength slits [25], Gao *et al.* [14] studied resonances by a rectangular cavity of mixed conducting parts. Using the MA methods, Joly and Tordeux [19–21] and Clausel *et al.* [11] studied resonances by thin slots, Holley and Schnitzer [17] studied resonances in a slab with a single slit, and Brandão *et al.* [8] studied resonances in a slab of finite conductivity with a single slit or periodic slits. Compared with 2D structures, three-dimensional (3D) subwavelength structures are more flexible in practical fabrication and can in fact realize resonators of higher quality factors [9,15,16,29]. Nevertheless, much fewer theories have been developed so far to rigorously study resonances in 3D structures. Resonances of acoustic waves in a three-dimensional slab of tiny circular or square holes have been studied in [13, 22].

In [38], we proposed a Fourier matching method (FMM) to study resonances in a slab of subwavelength slits. Unlike the existing methods on this topic, the FMM does not use Green's function of each slit, which is complicated. Instead, it takes advantage of the existence of a countable Fourier basis for each slit so that the scattering problem can be reformulated as a countable-dimensional linear system with a frequency-dependent coefficient matrix. Resonance are determined by studying when the coefficient matrix is non-invertible. However, the method relies on explicit asymptotics and delicate estimates of all elements of the matrix, largely limiting its extension to more general structures. To tackle this issue, this paper, using theories of functional analysis as our new tool, generalizes the FMM to analyze resonances in a 3D sound-hard slab with small cylindrical holes. The cross sections of the holes can be arbitrarily shaped. The new FMM completely avoids estimates of matrix elements and, as we shall see, works in a rather simple way. We would like to mention a recent work [24] of the first author, where the resonance theory for electromagnetic wave scattering in a nano annular gap was established based on a vectorial analogy of the FMM method.

Let $x=(x_1,x_2,x_3)$ denote a generic point in $\mathbb{R}^3$ and $x'=(x_1,x_2)$ be its projection in the x_1Ox_2-plane. As shown in Fig. 1, a sound-hard slab, occupying the rectangular region $\{x\in\mathbb{R}^3 : x_3\in(-l,0)\}$, has thickness l in x_3-direction, extends to infinity in both x_1- and x_2-directions and is embedded in a homogeneous medium of constant slowness. The slab contains N holes $\{V_{j,h}\}_{j=1}^N$, each of which has the same slowness as that of the surrounding medium. The holes $\{V_{j,h}\}_{j=1}^N$ for $j=1,\ldots,N$, satisfy the following conditions:

(i) $V_{j,h}$ is cylindrical, i.e. x_3-independent.

(ii) $V_{j,h}$ is generated by a Lipschitz domain $G_j\subset\mathbb{R}^2$ and a parameter $h\ll 1$ via

$$V_{j,h}:=\left\{x\in\mathbb{R}^3 : x'\in D_j+hG_j, x_3\in(-l,0)\right\},$$

where $\{D_j\}_{j=1}^N$ are well-separated points in $\mathbb{R}^2$, and $D_j+hG_j:=\{D_j+hx'\in\mathbb{R}^2 : x'\in G_j\}$ should be Lipschitz for $h\ll 1$. Therefore, $\{V_{j,h}\}_{j=1}^N$ of centers $\{C_j=(D_j,-l/2)\}_{j=1}^N$ are Lipschitz and well-separated for $h\ll 1$.

(iii) The area of G_j is l^2.

We note that condition (iii) is not necessary and is only introduced to simplify the presentation. In practice, G_j can be a star-shaped domain centered at $(0,0)$ so that D_j+hG_j resembles G_j.

In such a subwavelength structure, a sound field u of no external excitation is governed by

$$\Delta u+k^2u=0 \quad \text{on } \Omega_h, \tag{1.1}$$

$$\partial_\nu u=0 \quad \text{on } \partial\Omega_h, \tag{1.2}$$

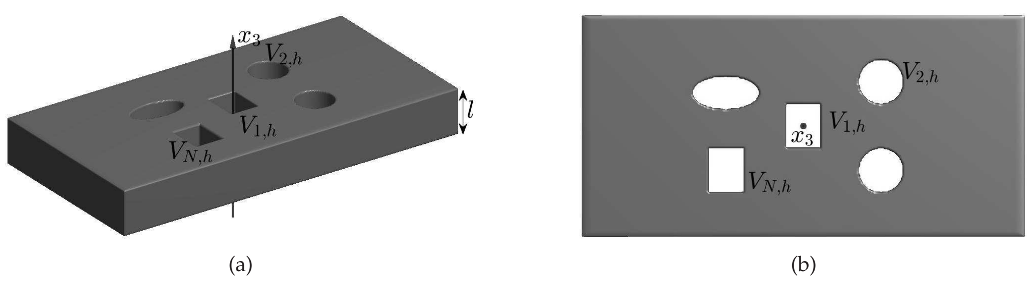

Figure 1: A sound-hard slab of thickness l with N small cylindrical holes $\{V_{j,h}\}_{j=1}^N$: (a) a side view; (b) a top view.

where the 3D Laplacian $\Delta=\sum_{i=1}^{3}\partial_{x_i}^2$, Ω_h is the interior of $\{x\in\mathbb{R}^3 : x_3\notin[-l,0]\}\cup(\cup_{j=1}^{N}\overline{V_{1,h}})$, $\partial\Omega_h$ denotes the boundary of Ω_h, ν denotes the outer unit normal vector along $\partial\Omega_h$, k denotes the frequency of u and for simplicity, the slowness in Ω_h is assumed to be 1. In the top region $\mathbb{R}^3_+=\{x\in\mathbb{R}^3 : x_3>0\}$, using Green's function for the Neumann boundary condition on $x_3=0$, we impose the following outgoing wave condition:

$$u(x)=-\int_{\Omega_h\cap\{x:x_3=0\}}\frac{e^{\mathbf{i}k|x-y|}}{2\pi|x-y|}\partial_{x_3}u(y)dS(y),\quad x_3>0. \tag{1.3}$$

This is equivalent to the usual Sommerfeld radiation condition in $\mathbb{R}^3_+$ when $k\in\mathbb{R}^+$ [4]. One may impose a similar outgoing wave condition in the bottom region $\{x\in\mathbb{R}^3 : x_3<-l\}$. This paper aims at locating the resonances of such a problem, i.e. the values of $k\in\mathbb{C}$ for which (1.1) and (1.2) possess nonzero outgoing solutions for $h\ll 1$.

By change of scale, l is assumed to be 1 from now on, so that k and h should be identified as kl and h/l, respectively, in general. We shall search k in a bounded region $\mathcal{B}=\{k\in\mathbb{C} : \mathrm{Re}(k)>0, |k|\in(\epsilon_0,K)\}$ for some sufficiently small constant $\epsilon_0>0$ and some sufficiently large constant $K>0$. Such a choice keeps k away from zero, a non-holomorphic point of the problem (a branch point of Eq. (2.4) for $\lambda_m=0$), and ensures $\epsilon:=kh\ll 1$, making regular asymptotic analysis for $h\ll 1$ applicable. Moreover, it is reasonable and of practical interests to consider sound waves away from the near-static regime and with wavelengths far greater than h, i.e. $2\pi/k\gg h$.

Due to the symmetry the structure, u can be assumed to be either evenly or oddly symmetric with respect to $x_3=-1/2$. Then, the original problem on Ω_h can be reformulated in the upper-half domain $\Omega_h^+=\Omega_h\cap\{x\in\mathbb{R}^3 : x_3>-1/2\}$ as

$$\begin{aligned}
&\Delta u+k^2u=0 && \text{on } \Omega_h^+,\\
&\partial_\nu u=0 && \text{on } \partial\Omega_h^+\backslash\overline{\cup_{j=1}^{N}\Gamma_{b;j,h}},\\
&\partial_\nu u=0 \text{ or } u=0 && \text{on } \cup_{j=1}^{N}\Gamma_{b;j,h},\\
&u \text{ satisfies the outgoing wave condition (1.3),}
\end{aligned}$$

where $\partial\Omega_h^+$ is the boundary of Ω_h^+ and

$$\Gamma_{b;j,h}=V_{j,h}\cap\left\{x\in\mathbb{R}^3 : x_3=-1/2\right\}.$$

Let

$$V_{j,h}^+=V_{j,h}\cap\left\{x\in\mathbb{R}^3 : x_3>-1/2\right\}$$

be the upper half of $V_{j,h}$ and

$$\Gamma_{j,h}=\partial V_{j,h}^+\cap\{x\in\mathbb{R}^3 : x_3=0\}$$

be the top boundary (aka the aperture) of $V_{j,h}^+$.

The new FMM goes as follows. In $\mathbb{R}^3_+$, u is expressed in terms of its normal derivatives on all the apertures $\{\Gamma_{j,h}\}_{j=1}^N$ according to (1.3). Inside each cylindrical hole $V^+_{j,h}$, we express u in terms of a countable Fourier basis due to the zero Neumann condition on the side surface of $V^+_{j,h}$. The countably infinite Fourier coefficients for all the holes constitute our targeted unknowns. Matching u on all the apertures $\{\Gamma_{j,h}\}_{j=1}^N$ yields a countably-dimensional linear system governing the unknowns. The system can be properly rescaled to make the new unknowns belong to the standard ℓ^2 space. Due to the invertibility of a principal submatrix of the infinite-dimensional coefficient matrix for $h \ll 1$, we are able to reduce the linear system to an N-dimensional linear system. Its unknowns correspond exactly to the N leading Fourier coefficients of u in the N holes $\{V^+_{j,h}\}_{j=1}^N$. Resonances can be determined by studying when the N-dimensional linear system is singular.

1.1 Fourier bases and equivalent norms of Sobolev spaces on the apertures

To separate the variables in u in each hole $V^+_{j,h}$, we need to find out a countable Fourier basis on each aperture $\Gamma_{j,h}$ first. This relies on the following generic result regarding spectral properties of the 2D Laplacian $\Delta_2 = \partial^2_{x_1} + \partial^2_{x_2}$ on the Lipschitz domain G_j with the zero Neumann boundary condition.

Theorem 1.1 ([30, Theorem 4.12])**.** *For the 2D Lipschitz domain $G_j, j=1,\ldots,N$, there exist sequences of functions $\phi_{1,j},\phi_{2,j},\cdots$ in $H^1(G_j)$, and of nonnegative numbers $\lambda_{0,j},\lambda_{1,j},\cdots$ having the following properties:*

(i) Each $\phi_{m,j}$ is an eigenfunction of $-\Delta_2$ in G_j with the eigenvalue $\lambda_{m,j}$

$$-\Delta_2 \phi_{m,j} = \lambda_{m,j}\phi_{m,j} \quad \text{on} \;\; G_j, \tag{1.4}$$

$$\partial_\nu \phi_{m,j} = 0 \qquad\qquad \text{on} \;\; \partial G_j. \tag{1.5}$$

(ii) The eigenvalues satisfy $0 = \lambda_{0,j} < \lambda_{1,j} \le \lambda_{2,j} \le \cdots$ with $\lambda_{m,j} \to \infty$ as $m \to \infty$.

(iii) The eigenfunctions $\{\phi_{m,j}\}_{m=0}^\infty$ form a complete orthonormal system, i.e. a Fourier basis, in $L_2(G_j)$; in particular, we can take $\phi_{0,j} = 1$.

(iv) For any $f \in H^1(G_j)$,

$$\|f\|^2_{H^1(G_j)} \eqsim \sum_{m=0}^{\infty} (1+\lambda_{m,j}) \big| (f,\phi_{m,j})_{L^2(G_j)} \big|^2.$$

Proof. Based on [30, Theorem 4.12], we only need to justify that $\lambda_{0,j}=0, \lambda_{1,j}>0$, and $\phi_{0,j}=1$. This can be seen that every eigenvalue $\lambda_{m,j}$ must be nonnegative by testing (1.4) with $\phi_{m,j}$. On the other hand, for $\lambda_{0,j}=0, \phi_{0,j}=1$ is the unique (up to a sign) and normalized (since the area of G_j is 1) solution of the eigenvalue problem so that $\lambda_{m,j}>0$ for $j \ge 1$. □

By Theorem 1.1(i)-(iii), the Laplacian $-\Delta_2$ on each aperture $\Gamma_{j,h}$ with the zero Neumann boundary condition has the eigenvalues $\{h^{-2}\lambda_{m,j}\}_{m=0}^{\infty}$. The associated eigenfunctions are

$$\left\{\psi_{m,j}^h(x'):=h^{-1}\phi_{m,j}\big((x'-D_j)/h\big)\right\}_{m=0}^{\infty}$$

forming a Fourier basis in $L^2(\Gamma_{j,h})$. Thus, for any $f\in L^2(\Gamma_{j,h})$, its Fourier coefficients form the following sequence:

$$\left\{f_{m,j}:=(f,\psi_{m,j}^h)_{L^2(\Gamma_{j,h})}\right\}_{m=0}^{\infty}$$

in ℓ^2 since by Parserval's identity,

$$\|f\|_{L^2(\Gamma_{j,h})}^2=\sum_{m=0}^{\infty}|f_{m,j}|^2=\big\|\{f_{m,j}\}_{m=0}^{\infty}\big\|_{\ell^2}^2.$$

On the other hand, Theorem 1.1(iv) indicates that for any $f\in H^1(\Gamma_{j,h})$,

$$\|f\|_{H^1(\Gamma_{j,h})}^2\approx\sum_{m=0}^{\infty}(1+\lambda_{m,j})|f_{m,j}|^2.$$

Now by the interpolation theory, $H^{1/2}(\Gamma_{j,h})$ can be equipped with the following norm:

$$\|f\|_{H^{1/2}(\Gamma_{j,h})}^2:=\sum_{m=0}^{\infty}(1+\lambda_{m,j})^{\frac{1}{2}}|f_{m,j}|^2,$$

so that $\tilde{H}^{-1/2}(\Gamma_{j,h})$, the dual space of $H^{1/2}(\Gamma_{j,h})$, can be equipped with

$$\|f\|_{\tilde{H}^{-1/2}(\Gamma_{j,h})}^2:=\sum_{m=0}^{\infty}(1+\lambda_{m,j})^{-\frac{1}{2}}|f_{m,j}|^2,$$

where $f_{m,j}$ should be understood as $\langle f,\psi_{m,j}^h\rangle_{\Gamma_{j,h}}$, a duality pairing on $\Gamma_{j,h}$.

The rest of this paper is organized as follows. In Section 2, we use the new FMM to study resonances in a sound-hard slab with a single hole, and to analyze the mechanism of field enhancement in the hole due to the resonances. In Section 3, following the single-hole framework of the FMM, we study resonances in a slab with multiple holes and provide general asymptotic formulae for the resonances. In Section 4, several numerical experiments are carried out to validate the proposed resonance formulae.

2 Resonances in a slab with a single hole

To clarify the basic idea, we begin with a sound-hard slab of thickness $l=1$ with a single hole, say, $V_{1,h}$ for $h\ll 1$. For simplicity, we assume $D_1=(0,0)$ so that $C_1=(0,0,-1/2)$ in this section. Thus, $N=1$ and Ω_h becomes the interior of $\{x\in\mathbb{R}^3: x_3\notin[-1,0]\}\cup\overline{V_{1,h}}$. As has been discussed, u can be assumed to be either evenly or oddly symmetric with respect to $x_3=-1/2$. They shall be respectively referred to as even or odd modes in the following. We first discuss the existence of nonzero even modes for $k\in\mathcal{B}$.

2.1 Even modes

Due to the symmetry, we find nonzero outgoing solutions $u\in H^1_{\rm loc}(\Omega_h^+)$ satisfying

$$\Delta u+k^2u=0 \quad \text{on } \Omega_h^+, \tag{2.1}$$
$$\partial_\nu u=0 \quad \text{on } \partial\Omega_h^+, \tag{2.2}$$

where

$$\Omega_h^+=\mathbb{R}^3_+\cup V_{1,h}^+\cup\Gamma_{1,h},\quad V_{1,h}^+=V_{1,h}\cap\{x\in\mathbb{R}^3: x_3\in(-1/2,0)\},$$

and $\Gamma_{1,h}$ is the aperture of $V_{1,h}^+$, as shown in Fig. 2.

Recall that the functions $\{\psi_{m,1}^h\}_{m=0}^\infty$ form a Fourier basis in $L^2(\Gamma_{1,h})$ and the corresponding Laplacian eigenvalues are $\{h^{-2}\lambda_{m,1}\}_{m=0}^\infty$. To simplify the presentation in Section 2, we shall suppress the hole index 1 in the previous symbols so that $V_h^+=V_{1,h}^+$, $\Gamma_h=\Gamma_{1,h},\Gamma_{b,h}=\Gamma_{b;1,h},G=G_1,\psi_m^h=\psi_{m,1}^h$ and $\lambda_m=\lambda_{m,1}$.

First, we use the Fourier basis $\{\psi_m^h\}_{m=0}^\infty$ to represent u in V_h^+. As $u|_{\Gamma_h}\in H^{1/2}(\Gamma_h)$, there exists a sequence $\{b_m\}_{m=0}^\infty$ such that

$$u|_{\Gamma_h}=\sum_{m=0}^\infty b_m\psi_m^h(e^{\mathbf{i}s_m}+1), \tag{2.3}$$

where

$$\begin{aligned} b_m&=(e^{\mathbf{i}s_m}+1)^{-1}(u|_{\Gamma_h},\psi_m^h)_{L^2(\Gamma_h)},\\ s_m&=h^{-1}\sqrt{\epsilon^2-\lambda_m}. \end{aligned} \tag{2.4}$$

Throughout this paper, the branch cut of $\sqrt{\cdot}$ is defined to be along the negative imaginary axis so that s_m of argument in $[-\pi/4,3\pi/4)$ becomes a holomorphic function of k in $\mathcal{B}$ for any $m\in\mathbb{N}$. Thus, $\{a_m:=\lambda_m^{1/4}b_m\}_{m>0}\in\ell^2$ for $h\ll 1$ since

$$\sum_{m=1}^\infty|a_m|^2\le C\sum_{m=0}^\infty(1+\lambda_m)^{\frac12}\left|b_m(e^{\mathbf{i}s_ml}+1)\right|^2=C\left\|u|_{\Gamma_h}\right\|^2_{H^{\frac12}(\Gamma_h)}<\infty.$$

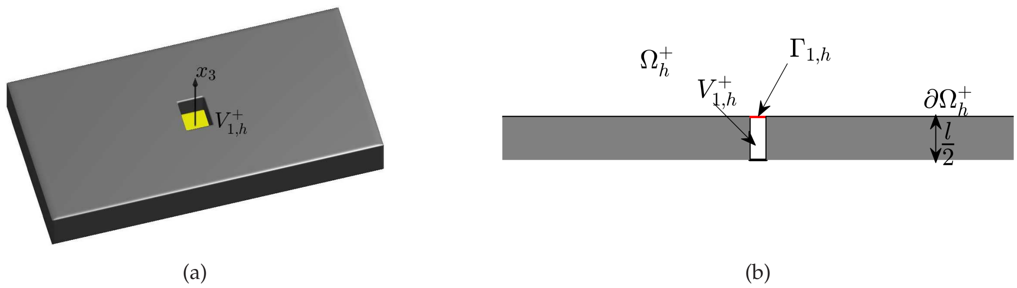

Figure 2: A slab with a single cavity $V_{1,h}^+$: (a) a top view; (b) the cross section at $x_1=0$.

Here and hereafter, we use $\{\cdot\}_{m>0}$ to denote a sequence starting with $m=1$. In V_h^+, let

$$u^-(x)=\sum_{m=0}^{+\infty} b_m\psi_m^h(x')\left[e^{\mathbf{i}s_m(x_3+1)}+e^{-\mathbf{i}s_m x_3}\right], \tag{2.5}$$

so that $\Phi:=u-u^-\in H^1(V_h^+)$ becomes a solution of

$$\begin{aligned} &\Delta\Phi=-k^2\Phi && \text{on } V_h^+,\\ &\partial_\nu\Phi=0 && \text{on } \partial V_h^+\backslash\overline{\Gamma_h},\\ &\Phi=0 && \text{on } \Gamma_h. \end{aligned}$$

When $k\in\mathcal{B}$ and $h\ll 1$, $-k^2$ is not an eigenvalue of the above problem so that $u=u^-$ on V_h^+. Therefore, the normal derivative of u on Γ_h becomes

$$\partial_\nu u=\partial_{x_3}u=\sum_{m=0}^{+\infty} b_m\mathbf{i}s_m\psi_m^h\left[e^{\mathbf{i}s_m}-1\right]\in\tilde{H}^{-\frac{1}{2}}(\Gamma_h). \tag{2.6}$$

If $e^{\mathbf{i}s_0}+1=0$, the representation (2.3) of $u|_{\Gamma_h}$ is impossible to determine b_0. To resolve this issue, we can use (2.6) to define $\{b_m\}_{m=0}^{\infty}$ so that the representation (2.5) becomes valid for all $k\in\mathcal{B}$ and $h\ll 1$.

In $\mathbb{R}_+^3$, the outgoing radiation condition (1.3) implies

$$u(x)=-\int_{\Gamma_h}\frac{e^{\mathbf{i}k|x-y|}}{2\pi|x-y|}\partial_{x_3}u(y)dS(y). \tag{2.7}$$

To match the two representations (2.5) and (2.7) on Γ_h, we need to study some properties of the following integral operators:

$$[\mathcal{S}\phi](x)=\int_{\Gamma_h}\frac{e^{\mathbf{i}k|x-y|}}{2\pi|x-y|}\phi(y)dS(y), \qquad x\in\Gamma_h, \tag{2.8}$$

$$[\mathcal{S}_0\phi](x)=\int_{\Gamma}\frac{1}{2\pi|x-y|}\phi(y)dS(y), \qquad x\in\Gamma, \tag{2.9}$$

$$[\mathcal{R}_0\phi](x)=\int_{\Gamma}\frac{e^{\mathbf{i}\epsilon|x-y|}-1-\mathbf{i}\epsilon|x-y|}{2\pi\epsilon^2|x-y|}\phi(y)dS(y), \quad x\in\Gamma, \tag{2.10}$$

where $\Gamma=\{x\in\mathbb{R}^3: x'\in G, x_3=0\}$ and we recall $\epsilon=kh\ll 1$.

Lemma 2.1. *For $k\in\mathcal{B}$ and $h\ll 1$, we have*

(i) $\mathcal{S}$ is bounded from $\tilde{H}^{-1/2}(\Gamma_h)$ to $H^{1/2}(\Gamma_h)$.

(ii) $\mathcal{S}_0$ is bounded from $\tilde{H}^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$, and is positive and bounded below, i.e. for any $\phi\in\tilde{H}^{-1/2}(\Gamma)$,

$$\langle\mathcal{S}_0\phi,\phi\rangle_\Gamma\geq C\|\phi\|_{\tilde{H}^{-1/2}(\Gamma)}$$

for some positive constant $C>0$.

(iii) $\mathcal{R}_0$ *is uniformly bounded from* $\tilde{H}^{-1/2}(\Gamma)$ *to* $H^{1/2}(\Gamma)$*, i.e.* $\|\mathcal{R}_0\| \le C$*, for some constant* $C>0$ *independent of* k *and* h*.*

Proof. Choose a bounded Lipschitz domain $\Omega_c \subset \mathbb{R}^3_+$ with its boundary Γ_c containing Γ. For any $\phi \in C^\infty_{\rm comp}(\Gamma), \mathcal{S}_0\phi = (2\mathcal{S}_c\phi)|_\Gamma$, where

$$[\mathcal{S}_c\phi](x) = \int_{\Gamma_c} G_0(x,y)\phi(y)dS(y)$$

with the 3D fundamental solution $G_0(x,y) = (4\pi|x-y|)^{-1}$ for the Laplacian Δ as the kernel. According to [30, Theorem 7.6, Corollary 8.13], $\mathcal{S}_c$ is bounded from $H^{-1/2}(\Gamma_c)$ to $H^{1/2}(\Gamma_c)$ and satisfies

$$\langle \mathcal{S}_c\phi,\phi\rangle_{\Gamma_c} \ge C/2\|\phi\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma)}$$

for some constant $C>0$ independent of ϕ. Thus, $\mathcal{S}_0$ is bounded from $\tilde{H}^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$, and

$$\langle \mathcal{S}_0\phi,\phi\rangle_\Gamma = \langle 2\mathcal{S}_c\phi,\phi\rangle_{\Gamma_c} \ge C\|\phi\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma)}.$$

The mapping property of $\mathcal{S}$ is similar. Now we prove (iii). Since the kernel of $\mathcal{R}_0$ in (2.10) and its gradient with respect to x are uniformly bounded for all $k \in \mathcal{B}$ and $h \ll 1$, we conclude that $\mathcal{R}_0$ is uniformly bounded from $L^2(\Gamma)$ to $H^1(\Gamma)$. The self-adjointness of $\mathcal{R}_0$ and the interpolation theory ensure the uniform boundedness of $\mathcal{R}_0$ from $\tilde{H}^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$. ◻

By (2.7), Lemma 2.1 implies that $u|_{\Gamma_h} = -\mathcal{S}\partial_{x_3}u \in H^{1/2}(\Gamma_h)$. By (2.3) and (2.6), the continuity of u and $\partial_\nu u$ on Γ_h yields

$$-\mathcal{S}\left[\sum_{m'=0}^{+\infty} b_{m'}\mathbf{i}s_{m'}\psi^h_{m'}\left(e^{\mathbf{i}s_{m'}}-1\right)\right] = \sum_{m'=0}^{\infty} b_{m'}\psi^h_{m'}\left(e^{\mathbf{i}s_{m'}}+1\right) \in H^{\frac{1}{2}}(\Gamma_h).$$

Based on the completeness of $\{\psi^h_m\}_{m=0}^\infty$, the above is equivalent to

$$-\sum_{m'=0}^{+\infty} b_{m'}\mathbf{i}s_{m'}hd_{m'm}\left(e^{\mathbf{i}s_{m'}}-1\right) = b_m[e^{\mathbf{i}s_m}+1]$$

for all $m \in \mathbb{N}$, where we have defined

$$d_{m'm} = h^{-1}\left(\mathcal{S}\psi^h_{m'},\psi^h_m\right)_{L^2(\Gamma_h)}. \tag{2.11}$$

The countably infinite number of equations can be properly rearranged in terms of the unknowns b_0 and $\{a_m\}_{m>0}$ as follows:

$$(e^{\mathbf{i}k}+1)b_0 = \mathbf{i}\epsilon(1-e^{\mathbf{i}k})d_{00}b_0 + \sum_{m'=1}^{\infty}\lambda_{m'}^{-\frac{1}{4}}\mathbf{i}s_{m'}h\left(1-e^{\mathbf{i}s_{m'}}\right)d_{m'0}a_{m'}, \tag{2.12}$$

$$\lambda_m^{-\frac{1}{4}}\left(e^{\mathbf{i}s_m}+1\right)a_m = \mathbf{i}\epsilon(1-e^{\mathbf{i}k})d_{0m}b_0 + \sum_{m'=1}^{\infty}\lambda_{m'}^{-\frac{1}{4}}\mathbf{i}s_{m'}h\left(1-e^{\mathbf{i}s_{m'}}\right)d_{m'm}a_{m'}. \tag{2.13}$$

Let $\psi_m^1=\psi_m^h|_{h=1}$. We have the following lemma regarding the asymptotic behavior of $d_{m'm}$ for $m,m'\in\mathbb{N}$.

Lemma 2.2. *For $k\in\mathcal{B}$ and $h\ll 1$,*

$$d_{m'm}=\begin{cases}(\mathcal{S}_0 1,1)_{L^2(\Gamma)}+\dfrac{\mathbf{i}\epsilon}{2\pi}+\mathcal{O}(\epsilon^2), & m=m'=0,\\ \big(\mathcal{S}_0\psi_{m'}^1,\psi_m^1\big)_{L^2(\Gamma)}+\epsilon^2\big(\mathcal{R}_0\psi_{m'}^1,\psi_m^1\big)_{L^2(\Gamma)}, & \text{otherwise},\end{cases} \tag{2.14}$$

where we recall that $\Gamma=\{x\in\mathbb{R}^3: x'\in G, x_3=0\}$.

Proof. By change of scale,

$$\begin{aligned}d_{m'm}&=\frac{1}{2\pi}\int_\Gamma\int_\Gamma\frac{e^{\mathbf{i}\epsilon|x-y|}}{|x-y|}\psi_{m'}^1(y')dS(y)\overline{\psi_m^1}(x')dS(x)\\&=\big(\mathcal{S}_0\psi_{m'}^1,\psi_m^1\big)_{L^2(\Gamma)}+\frac{\mathbf{i}\epsilon}{2\pi}\int_G\int_G\psi_{m'}^1(y')dy'\overline{\psi_m^1}(x')dx'+\epsilon^2\big(\mathcal{R}_0\psi_{m'}^1,\psi_m^1\big)_{L^2(\Gamma)}.\end{aligned}$$

It is clear that the second term of the right-hand side is nonzero only when $m=m'=0$. □

Now define for $m,m'\in\mathbb{N}^*$ that

$$\begin{aligned}c_{00}&=-\mathbf{i}\epsilon d_{00}, & c_{m'0}&=\lambda_{m'}^{-\frac14}\mathbf{i}s_{m'}h\big(1-e^{\mathbf{i}s_{m'}}\big)d_{m'0},\\ c_{0m}&=\lambda_m^{\frac14}\frac{-\mathbf{i}\epsilon}{e^{\mathbf{i}s_m}+1}d_{0m}, & c_{m'm}&=\lambda_{m'}^{-\frac14}\lambda_m^{\frac14}\mathbf{i}s_{m'}h\frac{1-e^{\mathbf{i}s_{m'}}}{e^{\mathbf{i}s_m}+1}d_{m'm}.\end{aligned}$$

We can rewrite (2.12) and (2.13) in the following vectorial form:

$$(1+e^{\mathbf{i}k})b_0=(e^{\mathbf{i}k}-1)c_{00}b_0+\big(\{a_{m'}\}_{m'>0},\{c_{m'0}\}_{m'>0}\big)_{\ell^2}, \tag{2.15}$$

$$(\mathcal{I}-\mathcal{A}_h)\{a_{m'}\}_{m'>0}=\{c_{0m}\}_{m>0}(e^{\mathbf{i}k}-1)b_0, \tag{2.16}$$

where $\mathcal{I}$ denotes the identity operator on ℓ^2, and the operator $\mathcal{A}_h:\ell^2\to\ell^2$ is defined as: For any $\{f_{m'}\}_{m'>0}\in\ell^2$,

$$\mathcal{A}_h\{f_{m'}\}_{m'>0}=\Big\{\sum_{m'>0}c_{m'm}f_{m'}\Big\}_{m>0}. \tag{2.17}$$

According to Lemma 2.2, the asymptotic behavior of $c_{m'm}$ for $m',m\in\mathbb{N}$ is given below.

Lemma 2.3. *For $k\in\mathcal{B}$ and $h\ll 1$*

1. $c_{00}=-(\mathcal{S}_0 1,1)_{L^2(\Gamma)}\epsilon\mathbf{i}+\dfrac{\epsilon^2}{2\pi}+\mathcal{O}(\epsilon^3)$.
2. *For $m\in\mathbb{N}^*$,*

$$c_{0m}=-\mathbf{i}\epsilon\Big(\mathcal{S}_0 1,\lambda_m^{\frac14}\psi_m^1\Big)_{L^2(\Gamma)}-\mathbf{i}\epsilon^3\Big(\mathcal{R}_0 1,\lambda_m^{\frac14}\psi_m^1\Big)_{L^2(\Gamma)}.$$

3. *For $m'\in\mathbb{N}^*$,*

$$c_{m'0}=-\left(\mathcal{S}_0\left[\lambda_{m'}^{\frac{1}{4}}\psi_{m'}^1\right],1\right)_{L^2(\Gamma)}-\epsilon^2\left(\mathcal{R}_0\left[\lambda_{m'}^{\frac{1}{4}}\psi_{m'}^1\right],1\right)_{L^2(\Gamma)}$$
$$+\left(\mathcal{S}_0\left[\lambda_{m'}^{-\frac{3}{4}}\psi_{m'}^1\right],1\right)\mathcal{O}(\epsilon^2).$$

4. *For $m',m\in\mathbb{N}^*$,*

$$c_{m'm}=-\left(\mathcal{S}_0\left[\lambda_{m'}^{\frac{1}{4}}\psi_{m'}^1\right],\lambda_m^{\frac{1}{4}}\psi_m^1\right)_{L^2(\Gamma)}-\epsilon^2\left(\mathcal{R}_0\left[\lambda_{m'}^{\frac{1}{4}}\psi_{m'}^1\right],\lambda_m\psi_m^1\right)_{L^2(\Gamma)}$$
$$+\left(\mathcal{S}_0\left[\lambda_{m'}^{-\frac{3}{4}}\psi_{m'}^1\right],\lambda_m^{\frac{1}{4}}\psi_m^1\right)_{L^2(\Gamma)}\mathcal{O}(\epsilon^2).$$

Therefore, $\{c_{0m}\}_{m>0}\in\ell^2,\{c_{m'0}\}_{m'>0}\in\ell^2$ and the operator $\mathcal{A}_h$ defined by (2.17) *is uniformly bounded from ℓ^2 to ℓ^2 for $k\in\mathcal{B}$ and $h\ll 1$. Moreover, $\mathcal{A}_h$ can be decomposed as $\mathcal{A}_h=\mathcal{P}+\epsilon^2\mathcal{Q}_h$, where $\mathcal{P}:\ell^2\to\ell^2$ is defined as: For any $\{f_{m'}\}_{m'>0}\in\ell^2$,*

$$\mathcal{P}\{f_{m'}\}_{m'>0}:=\left\{-\sum_{m'>0}\left(\mathcal{S}_0\left[\lambda_{m'}^{\frac{1}{4}}\psi_{m'}^1\right],\lambda_m^{\frac{1}{4}}\psi_m^1\right)_{L^2(\Gamma)}f_{m'}\right\}_{m>0}, \tag{2.18}$$

and $\mathcal{Q}_h:=\epsilon^{-2}(\mathcal{A}_h-\mathcal{P})$. Both $\mathcal{P}$ and $\mathcal{Q}_h$ are uniformly bounded from ℓ^2 to ℓ^2 for all $k\in\mathcal{B}$ and $h\ll 1$.

Proof. The asymptotics of $c_{m'm}$ for $m',m\in\mathbb{N}$ is straightforward by Lemma 2.2. As for the rest properties, we consider the case $h=0$ only, i.e. when the leading terms of $c_{m'm}$ are retained. For any two ℓ^2 sequences $\{f_m\}_{m>0}$ and $\{g_m\}_{m>0}$, the following two functions:

$$f=\sum_{m>0}f_m\lambda_m^{\frac{1}{4}}\psi_m^1,\quad g=\sum_{m>0}g_m\lambda_m^{\frac{1}{4}}\psi_m^1$$

are in $H^{-1/2}(\Gamma)$. Thus,

$$\begin{aligned}&\left|\sum_{m>0}\left(\mathcal{S}_0 1,\lambda_m^{\frac{1}{4}}\psi_m^1\right)_{L^2(\Gamma)}\overline{f}_m\right|\\&=|\langle\mathcal{S}_0 1,f\rangle_\Gamma|\le\|\mathcal{S}_0 1\|_{H^{1/2}(\Gamma)}\|f\|_{H^{-\frac{1}{2}}(\Gamma)}\\&\le C\|\{f_m\}_{m>0}\|_{\ell^2},\end{aligned}$$

so that $\{c_{m'0}\}_{m'>0}\in\ell^2$. Moreover,

$$\begin{aligned}&\left|(\mathcal{P}\{f_{m'}\}_{m'>0},\{g_m\}_{m>0})_{\ell^2}\right|\\&=\left|\sum_{m>0}\sum_{m'>0}\left(\mathcal{S}_0\left[\lambda_{m'}^{\frac{1}{4}}\psi_{m'}^1\right],\lambda_m^{\frac{1}{4}}\psi_m^1\right)_{L^2(\Gamma)}f_{m'}\overline{g_m}\right|\\&=|\langle\mathcal{S}_0 f,g\rangle|\le\|\mathcal{S}_0\|\cdot\|\{f_m\}_{m>0}\|_{\ell^2}\|\{g_m\}_{m>0}\|_{\ell^2},\end{aligned}$$

implying that $\mathcal{A}_h$ is bounded from ℓ^2 to ℓ^2. The case $0<h\ll 1$ can be analyzed similarly. The proof is complete. □

According to (2.18) and the positivity of $\mathcal{S}_0$ in Lemma 2.1, $\mathcal{I}-\mathcal{P}$ is strongly coercive on ℓ^2 since for any $\{f_m\}_{m>0}\in\ell^2$,

$$\begin{aligned}&\big((\mathcal{I}-\mathcal{P})\{f_{m'}\}_{m'>0},\{f_m\}_{m>0}\big)_{\ell^2}\\ =&\|\{f_{m'}\}_{m'>0}\|^2_{\ell^2}+\left\langle \mathcal{S}_0\left[\sum_{m'>0}\lambda_{m'}^{\frac{1}{4}}\psi^1_{m'}f_{m'}\right],\sum_{m>0}\lambda_m^{\frac{1}{4}}\psi^1_m f_m\right\rangle_{\Gamma_1}\\ \geq&\|\{f_{m'}\}_{m'>0}\|^2_{\ell^2}.\end{aligned}$$

Consequently, the Lax-Milgram theorem states that $\mathcal{I}-\mathcal{P}$ has a bounded inverse mapping from ℓ^2 to ℓ^2. Lemma 2.3 readily implies the invertibility of $\mathcal{I}-\mathcal{A}_h$ as shown below.

Theorem 2.1. *For $k\in\mathcal{B}$ and $h\ll 1$, $\mathcal{I}-\mathcal{A}_h$ has a uniformly bounded inverse mapping from ℓ^2 to ℓ^2. In fact,*

$$\big\|(\mathcal{I}-\mathcal{A}_h)^{-1}-(\mathcal{I}-\mathcal{P})^{-1}\big\|=\mathcal{O}(\epsilon^2).$$

Proof. By Lemma 2.3,

$$(\mathcal{I}-\mathcal{A}_h)^{-1}=\big(\mathcal{I}-\mathcal{P}+\epsilon^2\mathcal{Q}_h\big)^{-1}=(\mathcal{I}-\mathcal{P})^{-1}\big(\mathcal{I}+\epsilon^2\mathcal{Q}_h(\mathcal{I}-\mathcal{P})^{-1}\big)^{-1},$$

so that by the method of Neumann series,

$$\begin{aligned}&\big\|(\mathcal{I}-\mathcal{A}_h)^{-1}-(\mathcal{I}-\mathcal{P})^{-1}\big\|\\ =&\left\|(\mathcal{I}-\mathcal{P})^{-1}\sum_{n=1}^{\infty}\big(-\epsilon^2\mathcal{Q}_h(\mathcal{I}-\mathcal{P})^{-1}\big)^n\right\|\\ \leq&\epsilon^2\frac{\|(\mathcal{I}-\mathcal{P})^{-1}\|^2\|\mathcal{Q}_h\|}{1-\epsilon^2\big\|\mathcal{Q}_h(\mathcal{I}-\mathcal{P})^{-1}\big\|}=\mathcal{O}(\epsilon^2).\end{aligned}$$

The proof is complete. □

By Theorem 2.1, (2.15) and (2.16) are reduced to the following single equation:

$$\Big[(e^{\mathbf{i}k}+1)-(e^{\mathbf{i}k}-1)\Big(c_{00}+\big((\mathcal{I}-\mathcal{A}_h)^{-1}\{c_{0m}\}_{m>0},\{c_{m'0}\}_{m'>0}\big)_{\ell^2}\Big)\Big]b_0=0 \tag{2.19}$$

for the unknown b_0. The resonances for the even modes are characterized in the theorem below.

Theorem 2.2. *For any $h\ll 1$, the governing equations* (2.1) *and* (2.2) *possess nonzero solutions in $H^1_{\rm loc}(\Omega_h^+)$ for $k\in\mathcal{B}$, if and only if the following nonlinear equation of k:*

$$(e^{\mathbf{i}k}+1)-(e^{\mathbf{i}k}-1)\Big(c_{00}+\big((\mathcal{I}-\mathcal{A}_h)^{-1}\{c_{0m}\}_{m>0},\{c_{m'0}\}_{m'>0}\big)_{\ell^2}\Big)=0 \tag{2.20}$$

has solutions in $\mathcal{B}$. In fact, these solutions (the so-called resonances) are

$$k=k_{m,e}-2\mathbf{i}\Pi(\epsilon_{m,e})-4k_{m,e}^{-1}\Pi^2(\epsilon_{m,e})+\mathcal{O}\big(\epsilon_{m,e}^3\big),\quad m=1,2,\dots,M_e \tag{2.21}$$

for $\epsilon_{m,e}=k_{m,e}h\ll 1$*, where* $k_{m,e}=(2m-1)\pi$ *is a Fabry-Pérot frequency,*

$$\alpha=\left((\mathcal{I}-\mathcal{P})^{-1}\left\{\left(\mathcal{S}_0 1,\lambda_m^{\frac{1}{4}}\psi_m^1\right)_{L^2(\Gamma)}\right\}_{m>0},\left\{\left(\mathcal{S}_0 1,\lambda_m^{\frac{1}{4}}\psi_m^1\right)_{L^2(\Gamma)}\right\}_{m>0}\right)_{\ell^2}>0, \tag{2.22}$$

$$\Pi(\epsilon)=\left[-(\mathcal{S}_0 1,1)_{L^2(\Gamma)}+\alpha\right]\epsilon\mathbf{i}+\frac{\epsilon^2}{2\pi}, \tag{2.23}$$

and the positive integer M_e *is the greatest integer such that all the* M_e *frequencies lie in* $\mathcal{B}$*.*

Proof. By Theorem 2.1 and Lemma 2.3, Eq. (2.20) can be reduced to

$$e^{\mathbf{i}k}+1=(e^{\mathbf{i}k}-1)\Pi(\epsilon)+\mathcal{O}(\epsilon^3),$$

which is equivalent to

$$e^{\mathbf{i}k}+1=-\frac{2\Pi(\epsilon)}{1-\Pi(\epsilon)}+\mathcal{O}(\epsilon^3).$$

As the right-hand side approaches 0 as $\epsilon\to 0$, the resonances must satisfy: For some $m=1,\ldots,M_e,\delta_{m,e}:=k-k_{m,e}=o(1)$. Therefore, we have

$$e^{\mathbf{i}\delta_{m,e}}-1=\frac{2\Pi(\epsilon)}{1-\Pi(\epsilon)}+\mathcal{O}(\epsilon^3),$$

so that by Taylor's expansion of $\log(1+2x/(1-x))$ at $x=0$,

$$\begin{aligned}\delta_{m,e}&=-\mathbf{i}\log\left[1+\frac{2\Pi(\epsilon)}{1-\Pi(\epsilon)}+\mathcal{O}(\epsilon^3)\right]\\&=-2\mathbf{i}\Pi(\epsilon)+\mathcal{O}(\epsilon^3)\\&\approx 2\left[-(\mathcal{S}_0 1,1)_{L^2(\Gamma)}+\alpha\right]\epsilon_{m,e}.\end{aligned}$$

Thus, by the definition of Π,

$$\Pi(\epsilon)-\Pi(\epsilon_{m,e})=k_{m,e}^{-1}\delta_{m,e}\Pi(\epsilon_{m,e})+\mathcal{O}\left(\epsilon_{m,e}^3\right),$$

so that

$$\delta_{m,e}=-2\mathbf{i}\Pi(\epsilon_{m,e})-2\mathbf{i}k_{m,e}^{-1}\delta_{m,e}\Pi(\epsilon_{m,e})+\mathcal{O}\left(\epsilon_{m,e}^3\right).$$

Consequently,

$$\delta_{m,e}=\frac{-2\mathbf{i}\Pi(\epsilon_{m,e})+\mathcal{O}(\epsilon_{m,e}^3)}{1+2\mathbf{i}k_{m,e}^{-1}\Pi(\epsilon_{m,e})}=-2\mathbf{i}\Pi(\epsilon_{m,e})-4k_{m,e}^{-1}\Pi^2(\epsilon_{m,e})+\mathcal{O}\left(\epsilon_{m,e}^3\right).$$

From the above, we see that the resonances k, satisfying (2.20), must behave asymptotically as (2.21) for $\epsilon_{m,e}\ll 1$. As for the existence of such solutions, one notices that on the boundary of the disk $D_h=\{k\in\mathbb{C}:|k-k_{m,e}|\le h^{1/2}\}\subset\mathcal{B}$ for each $m\in 1,\ldots,M_e$,

$$\begin{aligned}&\left|(e^{\mathbf{i}k}+1)-(e^{\mathbf{i}k}-1)\left[c_{00}+\left((\mathcal{I}-\mathcal{A}_h)^{-1}\{c_{0m}\}_{m>0},\{c_{m'0}\}_{m'>0}\right)_{\ell^2}\right]+\mathbf{i}(k-k_{m,e})\right|\\&=\mathcal{O}(h)\le\sqrt{h}=|\mathbf{i}(k-k_{m,e})|.\end{aligned}$$

By Rouché's theorem, there exists a unique solution to (2.20) in D_h. □

2.2 Odd modes

Following the same procedure as in Section 2.1, we now study the resonances for the odd modes. We find a nonzero function $u\in H^1_{\rm loc}(\Omega_h^+)$ satisfying

$$\Delta u+k^2u=0 \quad \text{on } \Omega_h^+, \tag{2.24}$$

$$\partial_\nu u=0 \quad \text{on } \partial\Omega_h^+\backslash\overline{\Gamma_{b,h}}, \tag{2.25}$$

$$u=0 \quad \text{on } \Gamma_{b,h}, \tag{2.26}$$

where we recall that $\Gamma_{b,h}$ is the base of V_h^+.

In V_h^+, by the method of separation of variables, to satisfy (2.25) and (2.26), u is represented in terms of the Fourier basis $\{\psi_m^h\}_{m=0}^\infty$ as

$$u(x)=\sum_{m=0}^{+\infty}b_m\psi_m^h(x')\left[e^{\mathbf{i}s_m(x_3+1)}-e^{-\mathbf{i}s_mx_3}\right] \tag{2.27}$$

with the unknowns $\{b_m\}_{m=0}^\infty$ to be determined. Therefore, on Γ_h,

$$u=\sum_{m=0}^{+\infty}b_m\psi_m^h\left(e^{\mathbf{i}s_m}-1\right)\in H^{\frac{1}{2}}(\Gamma_h),$$

$$\partial_\nu u=\partial_{x_3}u=\sum_{m=0}^{+\infty}b_m\mathbf{i}s_m\psi_m^h\left(e^{\mathbf{i}s_m}+\right)\in \tilde{H}^{-\frac{1}{2}}(\Gamma_h),$$

implying that $\{a_m:=\lambda_m^{1/4}b_m\}_{m>0}\in\ell^2$.

Similar to the case of the even modes, the outgoing radiation condition (2.7) and the above two equations imply

$$-\mathcal{S}\left[\sum_{m'=0}^{+\infty}b_{m'}\mathbf{i}s_{m'}\psi_{m'}^h\left(e^{\mathbf{i}s_{m'}}+1\right)\right]=\sum_{m'=0}^{\infty}b_{m'}\psi_{m'}^h\left(e^{\mathbf{i}s_{m'}}-1\right)\in H^{\frac{1}{2}}(\Gamma_h).$$

The inner product of the above and ψ_m^h for any $m\in\mathbb{N}$ yields

$$(e^{\mathbf{i}k}-1)b_0=-\mathbf{i}\epsilon(e^{\mathbf{i}k}+1)d_{00}b_0-\sum_{m'=0}^{\infty}\lambda_{m'}^{-\frac{1}{4}}\mathbf{i}s_{m'}h\left(e^{\mathbf{i}s_{m'}}+1\right)d_{m'0}a_{m'}, \tag{2.28}$$

$$\lambda_m^{-\frac{1}{4}}\left(1-e^{\mathbf{i}s_m}\right)a_m=\mathbf{i}\epsilon\left(e^{\mathbf{i}s_0}+1\right)d_{0m}b_0+\sum_{m'=1}^{\infty}\lambda_{m'}^{-\frac{1}{4}}\mathbf{i}s_{m'}h\left(e^{\mathbf{i}s_{m'}}+1\right)d_{m'm}a_{m'} \tag{2.29}$$

for the unknowns b_0 and $\{a_m\}_{m>0}$, where we recall that $\{d_{m'm}\}_{m,m'=0}^\infty$ have been defined in (2.11) and $\epsilon=kh$. Now define

$$c_{00}^o=-\mathbf{i}\epsilon d_{00}, \qquad c_{m'0}^o=-\lambda_{m'}^{-\frac{1}{4}}\mathbf{i}s_{m'}h\left(1+e^{\mathbf{i}s_{m'}}\right)d_{m'0},$$

$$c_{0m}^o=\lambda_m^{\frac{1}{4}}\frac{\mathbf{i}\epsilon}{1-e^{\mathbf{i}s_m}}d_{0m}, \qquad c_{m'm}^o=\lambda_{m'}^{-\frac{1}{4}}\lambda_m^{\frac{1}{4}}\mathbf{i}s_{m'}h\frac{1+e^{\mathbf{i}s_{m'}}}{1-e^{\mathbf{i}s_m}}d_{m'm}$$

for $m,m'\in\mathbb{N}^*$. We rewrite (2.28) and (2.29) in the following vectorial form:

$$(e^{\mathbf{i}k}-1)b_0=(e^{\mathbf{i}k}+1)c^o_{00}b_0+\left(\{a_{m'}\}_{m'>0},\{c^o_{m'0}\}_{m'>0}\right)_{\ell^2}, \tag{2.30}$$
$$\{a_m\}_{m>0}=b_0(e^{\mathbf{i}k}+1)\{c^o_{0m}\}_{m>0}+\mathcal{A}^o_h\{a_{m'}\}_{m'>0}, \tag{2.31}$$

where the operator $\mathcal{A}^o_h:\ell^2\to\ell^2$ is defined as: For any $\{f_{m'}\}_{m'>0}\in\ell^2$,

$$\mathcal{A}^o_h\{f_{m'}\}_{m'>0}=\left\{\sum_{m'>0}c^o_{m'm}f_{m'}\right\}_{m>0}. \tag{2.32}$$

Similar to Lemma 2.3, we obtain the asymptotic behavior of $c^o_{m'm}$ for $m',m\in\mathbb{N}$.

Lemma 2.4. *For $k\in\mathcal{B}$ and $h\ll 1$*

1. $c^o_{00}=-(\mathcal{S}_0 1,1)_{L^2(\Gamma)}\epsilon\mathbf{i}+\dfrac{\epsilon^2}{2\pi}+\mathcal{O}(\epsilon^3)$.
2. *For $m\in\mathbb{N}^*$,*
$$c^o_{0m}=\mathbf{i}\epsilon\left(\mathcal{S}_0 1,\lambda_m^{\frac14}\psi^1_m\right)_{L^2(\Gamma)}+\mathbf{i}\epsilon^3\left(\mathcal{R}_0 1,\lambda_m^{\frac14}\psi^1_m\right)_{L^2(\Gamma)}.$$
3. *For $m'\in\mathbb{N}^*$,*
$$\begin{aligned}c^o_{m'0}=&\left(\mathcal{S}_0\left[\lambda_{m'}^{\frac14}\psi^1_{m'}\right],1\right)_{L^2(\Gamma)}+\epsilon^2\left(\mathcal{R}_0\left[\lambda_{m'}^{\frac14}\psi^1_{m'}\right],1\right)_{L^2(\Gamma)}\\&+\left(\mathcal{S}_0\left[\lambda_{m'}^{-\frac34}\psi^1_{m'}\right],1\right)_{L^2(\Gamma)}\mathcal{O}(\epsilon^2).\end{aligned}$$
4. *For $m',m\in\mathbb{N}^*$,*
$$\begin{aligned}c^o_{m'm}=&-\left(\mathcal{S}_0\left[\lambda_{m'}^{\frac14}\psi^1_{m'}\right],\lambda_m^{\frac14}\psi^1_m\right)_{L^2(\Gamma)}-\epsilon^2\left(\mathcal{R}_0\left[\lambda_{m'}^{\frac14}\psi^1_{m'}\right],\lambda_m^{\frac14}\psi^1_m\right)_{L^2(\Gamma)}\\&+\left(\mathcal{S}_0\left[\lambda_{m'}^{-\frac34}\psi^1_{m'}\right],\lambda_m^{\frac14}\psi^1_m\right)_{L^2(\Gamma)}\mathcal{O}(\epsilon^2).\end{aligned}$$

Therefore, $\{c^o_{0m}\}_{m>0}\in\ell^2,\{c^o_{m'0}\}_{m'>0}\in\ell^2$, and the operator $\mathcal{A}^o_h$ defined in (2.32) *is uniformly bounded from ℓ^2 to ℓ^2 for all $k\in\mathcal{B}$ and $h\ll 1$. Moreover, $\mathcal{A}^o_h$ can be decomposed as $\mathcal{A}^o_h=\mathcal{P}+\epsilon^2\mathcal{Q}^o_h$, where $\mathcal{Q}^o_h:=\epsilon^{-2}(\mathcal{A}^o_h-\mathcal{P})$ is uniformly bounded from ℓ^2 to ℓ^2.*

By analogy with Theorem 2.1, we obtain next theorem.

Theorem 2.3. *For $k\in\mathcal{B}$ and $h\ll 1,\mathcal{I}-\mathcal{A}^o_h$ has a uniformly bounded inverse mapping from ℓ^2 to ℓ^2. In fact,*

$$\left\|\left(\mathcal{I}-\mathcal{A}^o_h\right)^{-1}-(\mathcal{I}-\mathcal{P})^{-1}\right\|=\mathcal{O}(\epsilon^2).$$

Based on Theorem 2.3, (2.30) and (2.31) are reduced to the following single equation:

$$\left[(e^{\mathbf{i}k}-1)-(e^{\mathbf{i}k}+1)\left(c^o_{00}+\left(\left(\mathcal{I}-\mathcal{A}^o_h\right)^{-1}\left\{c^o_{0m}\right\}_{m>0},\left\{c^o_{m'0}\right\}_{m'>0}\right)_{\ell^2}\right)\right]b_0=0 \tag{2.33}$$

for the unknown b_0. The resonances for the odd modes are characterized as follows.

Theorem 2.4. *For any $h\ll 1$, the governing equations* (2.24)-(2.26) *possess nonzero solutions in $H^1_{\rm loc}(\Omega^+_h)$ for $k\in\mathcal{B}$, if and only if the following nonlinear equation of k:*

$$(e^{\mathbf{i}k}-1)-(e^{\mathbf{i}k}+1)\left(c^o_{00}+\left(\left(\mathcal{I}-\mathcal{A}^o_h\right)^{-1}\left\{c^o_{0m}\right\}_{m>0},\left\{c^o_{m'0}\right\}_{m'>0}\right)_{\ell^2}\right)=0 \tag{2.34}$$

has solutions in $\mathcal{B}$. In fact, these solutions (the so-called resonances) are

$$k=k_{m,o}-2\mathbf{i}\Pi(\epsilon_{m,o})-4k^{-1}_{m,o}\Pi^2(\epsilon_{m,o})+\mathcal{O}\left(\epsilon^3_{m,o}\right),\quad m=1,2,\ldots,M_o \tag{2.35}$$

for $\epsilon_{m,o}=k_{m,o}h\ll 1$, where $k_{m,o}=2m\pi$ is a Fabry-Pérot frequency and the positive integer M_o is the greatest integer such that all the M_o frequencies lie in $\mathcal{B}$.

Proof. By Theorem 2.3 and Lemma 2.4, Eq. (2.20) can be reduced to

$$e^{\mathbf{i}k}-1=(e^{\mathbf{i}k}+1)\Pi(\epsilon)+\mathcal{O}(\epsilon^3),$$

which is equivalent to

$$e^{\mathbf{i}k}-1=\frac{2\Pi(\epsilon)}{1-\Pi(\epsilon)}+\mathcal{O}(\epsilon^3).$$

Since the right-hand side approaches 0 as $\epsilon\to 0$, the resonances must satisfy: For some $m=1,\ldots,M_o,\delta_{m,o}:=k-k_{m,o}=o(1)$. Note that $k\neq 0$ for $0\notin\mathcal{B}$ and that $\liminf_{h\to 0}|k|>0$ since

$$\lim_{h\to 0}\frac{e^{\mathbf{i}k}-1}{k}=\lim_{h\to 0}k^{-1}\left[\frac{2\Pi(\epsilon)}{1-\Pi(\epsilon)}+\mathcal{O}(\epsilon^3)\right]=0.$$

The rest proof follows from arguments similar to that in the proof of Theorem 2.2. □

2.3 Field enhancement

To conclude Section 2, we study the mechanism of field enhancement in the hole V_h by the proposed FMM. For simplicity, let a normal incident wave field $u^{\rm inc}=e^{-\mathbf{i}k_0x_3}$ of frequency $k_0>0$ be specified in $\mathbb{R}^3_+$, where k_0 equals the real part of some resonances given by (2.21) or (2.35). The scattering problem in Ω_h can be reduced by symmetry to two subproblems defined in Ω^+_h: By specifying the incident field $u^{\rm inc}/2$ in $\mathbb{R}^3_+$, we solve Eqs. (2.1) and (2.2) for an evenly symmetric total field u with respect to $x_3=-1/2$, or Eqs. (2.24)-(2.26) for an oddly symmetric total field u with respect to $x_3=-1/2$. Theorems 2.2 and 2.4 in fact have justified the well-posedness of the two subproblems for $h\ll 1$ since any real frequency k_0 is not a resonance. The solution to the original problem is the sum of the two different symmetric wave fields. We consider problem (ii) in the following, problem (i) can be analyzed similarly. In $\mathbb{R}^3_+$, define

$$u^{\mathrm{ref}}(x):=\frac{u^{\mathrm{inc}}(x',x_3)}{2}+\frac{u^{\mathrm{inc}}(x',-x_3)}{2}=\cos(k_0x_3). \tag{2.36}$$

Then, $u-u^{\mathrm{ref}}$ satisfies the outgoing radiation condition (1.3). Following the same procedures in Section 2.2, we obtain the following inhomogeneous equation:

$$-\mathcal{S}\partial_{x_3}u+u^{\mathrm{ref}}=\sum_{m'=0}^{\infty}b_{m'}\psi^h_{m'}\left[e^{\mathbf{i}s_{m'}}-1\right] \quad \text{on } \Gamma_h, \tag{2.37}$$

where $\mathcal{S}$ and $s_{m'}$ are defined in the same way but with k replaced by k_0. The inner product of (2.37) and each ψ^h_m for $m\in\mathbb{N}$ yields

$$\left(e^{\mathbf{i}k_0}-1\right)b_0=\left(e^{\mathbf{i}k_0}+1\right)c^o_{00}b_0+\left(\{a_{m'}\}_{m'>0},\left\{c^o_{m'0}\right\}_{m'>0}\right)_{\ell^2}+b^{\mathrm{ref}}_0, \tag{2.38}$$

$$\{a_m\}_{m>0}=\left\{c^o_{0m}\right\}_{m>0}\left(e^{\mathbf{i}k_0}+1\right)b_0+\mathcal{A}^o_h\{a_m\}_{m>0}+\left\{a^{\mathrm{ref}}_m\right\}_{m>0}, \tag{2.39}$$

where

$$b^{\mathrm{ref}}_0:=\int_{\Gamma_h}u^{\mathrm{ref}}(x)\overline{\psi^h_0(x')}dS(x)=h,$$

$$a^{\mathrm{ref}}_m:=\frac{\lambda_m^{\frac{1}{4}}}{e^{\mathbf{i}s_m}-1}\int_{\Gamma_h}u^{\mathrm{ref}}(x)\overline{\psi^h_m(x')}dS(x)=0.$$

For $h\ll 1$, Theorem 2.3 implies that (2.38) and (2.39) have the following unique solution:

$$b_0=\frac{h}{\left[\left(e^{\mathbf{i}k_0}-1\right)-\left(e^{\mathbf{i}k_0}+1\right)\Pi(k_0h)+\mathcal{O}\left(k_0^3h^3\right)\right]}, \tag{2.40}$$

$$\{a_m\}=\left(\mathcal{I}-\mathcal{A}^{(o)}_h\right)^{-1}\left\{c^{(o)}_{0m}\right\}_{m>0}\left(e^{\mathbf{i}k_0}+1\right)b_0. \tag{2.41}$$

For $k_0=\mathrm{Re}(k)$ with k defined in (2.35) for some integer $m\in[1,M_o]$,

$$k-k_0=\mathrm{Im}(k)\mathbf{i}=-\frac{\mathbf{i}\epsilon^2_{m,o}}{\pi}+\mathcal{O}(\epsilon^3_{m,o}),$$

so that

$$e^{\mathbf{i}k_0}-e^{\mathbf{i}k}=e^{\mathbf{i}k_0}\left(1-e^{\mathbf{i}(k-k_0)}\right)=\left(-\mathbf{i}(k-k_0)\right)+\mathcal{O}\left(\epsilon^3_{m,o}\right)=\frac{-\epsilon^2_{m,o}}{\pi}+\mathcal{O}\left(\epsilon^3_{m,o}\right),$$

$$\Pi(k_0h)-\Pi(kh)=\mathcal{O}\left(\epsilon^3_{m,o}\right).$$

Thus,

$$\begin{aligned}&\left(e^{\mathbf{i}k_0}-1\right)-\left(e^{\mathbf{i}k_0}+1\right)\Pi(k_0h)\\&=(e^{\mathbf{i}k}-1)-(e^{\mathbf{i}k}+1)\Pi(kh)+\frac{-\epsilon^2_{m,o}}{\pi}+\mathcal{O}\left(\epsilon^3_{m,o}\right)\\&=\frac{-\epsilon^2_{m,o}}{\pi}+\mathcal{O}\left(\epsilon^3_{m,o}\right),\end{aligned}$$

so that

$$b_0=\frac{-\pi}{k_{m,o}\epsilon_{m,o}}+\mathcal{O}(1),\quad \|\{a_m\}_{m>0}\|_{\ell^2}=\mathcal{O}(1).$$

Inside the hole V_h^+,

$$\begin{aligned}u(x)&=\frac{1}{h}b_0\left[e^{\mathbf{i}k_0(x_3+1)}-e^{-\mathbf{i}k_0x_3}\right]+\sum_{m=1}^{\infty}a_m\lambda_m^{-\frac{1}{4}}\psi_m^h(x')\left[e^{\mathbf{i}s_m(x_3+1)}-e^{-\mathbf{i}s_mx_3}\right]\\&=\frac{-2\pi\mathbf{i}}{\epsilon_{m,o}^2}\sin(k_0x_3)+\mathcal{O}(h^{-1})-\sum_{m=1}^{\infty}a_m\lambda_m^{-\frac{1}{4}}\psi_m^h(x')e^{-\mathbf{i}s_mx_3}+\mathcal{O}\left(e^{-\frac{\lambda_1}{2h}}\right),\end{aligned}$$

where the prefactors in the $\mathcal{O}$-terms depend on m and we have used the fact that $e^{\mathbf{i}k_0}-1=\mathcal{O}(h)$. Consequently,

$$u(x)=\begin{cases}\mathcal{O}(h^{-1}) & \text{for}\quad h^2\ll x_3\ll h,\\ \dfrac{-2\pi\mathbf{i}}{\epsilon_{m,o}^2}\sin(k_0x_3)+\mathcal{O}(h^{-1}) & \text{for a fixed}\ \ x_3\in(-1/2,0)\end{cases}$$

exhibiting the field enhancement of different magnitudes near the aperture and inside the hole. It is interesting to notice that the leading field pattern of u in the hole V_h^+ is independent of the shape of G.

3 Multiple holes

In this section, we study resonances in a slab containing the N holes $\{V_{j,h}\}_{j=1}^N$ defined in the introduction. As in [38], we start with the case of two holes to clarify the basic idea.

3.1 Two holes

Assume $N=2$. As mentioned before, the problem can be reduced by symmetry to two half-space problems on the domain Ω_h^+. We here study the resonances for the even modes only and shall directly show the results for the odd modes. Let $\Gamma_{A,h}:=\cup_{j=1}^2\Gamma_{j,h}$, and $\Gamma_j:=G_j\times\{x_3=0\}$ for $j=1,2$.

We find $k\in\mathcal{B}$ such that there exists a nonzero $u\in H^1_{\rm loc}(\Omega_h^+)$ satisfying

$$\Delta u+k^2u=0 \quad \text{on}\ \ \Omega_h^+, \tag{3.1}$$

$$\partial_\nu u=0 \quad \text{on}\ \ \partial\Omega_h^+. \tag{3.2}$$

Recall that for $j=1,2$, the eigenfunctions $\{\psi_{m,j}^h\}_{m=0}^\infty$ constitute a Fourier basis in $L^2(\Gamma_{j,h})$ and $\{h^{-2}\lambda_{m,j}\}_{m=0}^\infty$ are the associated Laplacian eigenvalues. In either hole $V_{j,h}^+$, the sound field u is expressed by

$$u(x)=\sum_{m'=0}^{+\infty}b_{m',j}\psi_{m',j}^h(x')\left[e^{\mathbf{i}s_{m',j}(x_3+1)}+e^{-\mathbf{i}s_{m',j}x_3}\right], \tag{3.3}$$

where

$$s_{m',j}=h^{-1}\sqrt{\epsilon^2-\lambda_{m',j}},$$

and $\{b_{m',j}\}_{m'=0}^{\infty}$ are the unknowns to be determined. The normal derivative on $\Gamma_{A,h}$ becomes

$$\partial_\nu u|_{\Gamma_{j,h}}:=\partial_{x_3}u|_{\Gamma_{j,h}}=\sum_{m=0}^{+\infty}b_{m,j}\mathbf{i}s_{m,j}\psi_{m,j}^h\left[e^{\mathbf{i}s_{m,j}}-1\right], \tag{3.4}$$

so that the sequence $\{a_{m,j}:=\lambda_{m,j}^{1/4}b_{m,j}\}\in\ell^2$ for $j=1,2$. The outgoing radiation condition (1.3) in $\mathbb{R}_+^3$ for $N=2$ becomes

$$u(x)=-\sum_{j=1}^{2}\int_{\Gamma_{j,h}}\frac{e^{\mathbf{i}k|x-y|}}{2\pi|x-y|}\sum_{m=0}^{+\infty}b_{m,j}\mathbf{i}s_{m,j}\psi_{m,j}^h(y')\left[e^{\mathbf{i}s_{m,j}}-1\right]dS(y)\in H^1_{\rm loc}(\mathbb{R}^3_+).$$

Now define for any $x\in\Gamma_{A,h}$ that

$$[\mathcal{S}_j\phi](x)=\int_{\Gamma_{j,h}}\frac{e^{\mathbf{i}k|x-y|}}{2\pi|x-y|}\phi(y)dS(y). \tag{3.5}$$

It can be seen that $\mathcal{S}_j$ is bounded from $\tilde{H}^{-1/2}(\Gamma_{j;h})$ to $H^{1/2}(\Gamma_{A,h})$ for $j=1,2$. On $\Gamma_{A,h}$,

$$u=-\big(\mathcal{S}_1[\partial_\nu u|_{\Gamma_{1,h}}]+\mathcal{S}_2[\partial_\nu u|_{\Gamma_{2,h}}]\big),$$

so that the continuity of u implies

$$-\big(\mathcal{S}_1[\partial_\nu u|_{\Gamma_{1,h}}]+\mathcal{S}_2[\partial_\nu u|_{\Gamma_{2,h}}]\big)\big|_{\Gamma_{j,h}}=\sum_{m'=0}^{+\infty}b_{m',j}\psi_{m',j}^h\big(e^{\mathbf{i}s_{m',j}}+1\big),\quad j=1,2.$$

They are equivalent to

$$\Big(-\big(\mathcal{S}_1[\partial_\nu u|_{\Gamma_{1,h}}]+\mathcal{S}_2[\partial_\nu u|_{\Gamma_{2,h}}]\big)\big|_{\Gamma_{j,h}},\psi_{m,j}^h\Big)_{L^2(\Gamma_{j,h})}=b_{m,j}\big(e^{\mathbf{i}s_{m,j}}+1\big),\quad j=1,2,\quad m\in\mathbb{N}. \tag{3.6}$$

For $i,j=1,2$ and $m,m'\in\mathbb{N}$, let

$$d_{m'm}^{ij}=h^{-1}\big(\big[\mathcal{S}_i\psi_{m',i}^h\big]\big|_{\Gamma_{j,h}},\psi_{m,j}^h\big)_{L^2(\Gamma_{j,h})}, \tag{3.7}$$

and

$$c_{m'm}^{ij}=\begin{cases}-\mathbf{i}\epsilon d_{00}^{ij}, & m'=m=0,\\ \lambda_{m',i}^{-\frac14}\mathbf{i}s_{m',i}h(1-e^{\mathbf{i}s_{m',i}})d_{m'0}^{ij}, & m'>0=m,\\ \lambda_{m,j}^{\frac14}\dfrac{-\mathbf{i}\epsilon}{e^{\mathbf{i}s_{m,j}}+1}d_{0m}^{ij}, & m>0=m',\\ \lambda_{m',i}^{-\frac14}\lambda_{m,j}^{\frac14}\mathbf{i}s_{m',i}h\dfrac{1-e^{\mathbf{i}s_{m',i}}}{e^{\mathbf{i}s_{m,j}}+1}d_{m'm}^{ij}, & m,m'>0.\end{cases}$$

By analogy to the Eqs. (2.15) and (2.16) in Section 2.1, the countable number of equations in (3.6) can be rewritten in terms of matrix operators as follows:

$$(e^{\mathbf{i}k}+1)\begin{bmatrix} b_{0,1} \\ b_{0,2} \end{bmatrix} = (e^{\mathbf{i}k}-1)\begin{bmatrix} c_{00}^{11} & c_{00}^{21} \\ c_{00}^{12} & c_{00}^{22} \end{bmatrix}\begin{bmatrix} b_{0,1} \\ b_{0,2} \end{bmatrix} + \begin{bmatrix} \{c_{m'0}^{11}\} & \{c_{m'0}^{21}\} \\ \{c_{m'0}^{12}\} & \{c_{m'0}^{22}\} \end{bmatrix}\begin{bmatrix} \{a_{m',1}\} \\ \{a_{m',2}\} \end{bmatrix}, \tag{3.8}$$

$$\begin{bmatrix} \mathcal{I} & \\ & \mathcal{I} \end{bmatrix}\begin{bmatrix} \{a_{m,1}\} \\ \{a_{m,2}\} \end{bmatrix} = (e^{\mathbf{i}k}-1)\begin{bmatrix} \{c_{0m}^{11}\} & \{c_{0m}^{21}\} \\ \{c_{0m}^{12}\} & \{c_{0m}^{22}\} \end{bmatrix}\begin{bmatrix} b_{0,1} \\ b_{0,2} \end{bmatrix} + \begin{bmatrix} \mathcal{A}_h^{11} & \mathcal{A}_h^{21} \\ \mathcal{A}_h^{12} & \mathcal{A}_h^{22} \end{bmatrix}\begin{bmatrix} \{a_{m',1}\} \\ \{a_{m',2}\} \end{bmatrix} \tag{3.9}$$

for the unknowns $b_{0,j}$ and $\{a_{m,j}\}, j=1,2$. In the above, all the bracketed ℓ^2 sequences start with $m=1$ or $m'=1$, and the operators $\mathcal{A}_h^{ij}$ are defined by (2.17) but with $c_{m'm}$ replaced by $c_{m'm}^{ij}$. Lemma 2.2 describes the asymptotic behavior of $d_{m'm}^{ij}$ for $i=j$ by replacing the symbols Γ and ψ_m^h by Γ_j and $\psi_{m,j}^h$, respectively. If $i\neq j$, the asymptotic behavior of $d_{m'm}^{ij}$ is shown below.

Lemma 3.1. *Let $i\in\{1,2\}$ and $j=3-i$. For $k\in\mathcal{B}$ and $h\ll 1$,*

$$d_{m'm}^{ij} = \begin{cases} \dfrac{e^{\mathbf{i}k|C_{ij}|}}{2\pi k|C_{ij}|}\epsilon+\mathcal{O}(\epsilon^2), & m=m'=0, \\ \epsilon^2\left(\mathcal{R}_0^{ij}\psi_{m',i}^1,\psi_{m,j}^1\right)_{L^2(\Gamma_j)}, & \text{otherwise}, \end{cases} \tag{3.10}$$

where $C_{ij}=C_i-C_j, \psi_{m,j}^1=\psi_{m,j}^h|_{h=1}$, and $\mathcal{R}_0^{ij}$ is a uniformly bounded operator from $\tilde{H}^{-1/2}(\Gamma_i)$ to $H^{1/2}(\Gamma_j)$.

Proof. We consider $i=2$ and $j=1$ only. By change of scale,

$$\begin{aligned} d_{m'm}^{21} &= \frac{1}{2\pi}\int_{\Gamma_2}\int_{\Gamma_1}\frac{he^{\mathbf{i}k|h(x-y)-C_{21}|}}{|h(x-y)-C_{21}|}\psi_{m,1}^1(y')dS(y)\psi_{m',2}^1(x')dS(x) \\ &= \frac{\epsilon e^{\mathbf{i}k|C_{21}|}}{2\pi k|C_{21}|}\int_{\Gamma_2}\int_{\Gamma_1}\psi_{m,1}^1(y')dS(y)\psi_{m',2}^1(x')dS(x) \\ &\quad +\frac{\epsilon^2}{2\pi}\int_{\Gamma_2}\int_{\Gamma_1}\left[\frac{e^{\mathbf{i}k|h(x-y)-C_{21}|}}{\epsilon|\epsilon(x-y)-kC_{21}|}-\frac{e^{\mathbf{i}k|C_{21}|}}{\epsilon k|C_{21}|}\right]\psi_{m,1}^1(y')dS(y)\psi_{m',2}^1(x')dS(x). \end{aligned}$$

Eq. (3.10) follows from that the first integral on the right-hand side of the above is nonzero only when $m=m'=0$, and that the second integral (excluding the prefactor $\epsilon^2/(2\pi)$) has a smooth kernel, which and the gradient of which are uniformly bounded for $(x,y)\in\Gamma_2\times\Gamma_1$. ∎

As an immediate consequence, we get the following lemma.

Lemma 3.2. *Let $i\in\{1,2\}$ and $j=3-i$. For $k\in\mathcal{B}$ and $h\ll 1$,*

$$c^{ij}_{m'm}=\begin{cases}-\dfrac{\mathbf{i}e^{\mathbf{i}k|C_{ij}|}}{2\pi k|C_{ij}|}\epsilon^2+\mathcal{O}(\epsilon^3), & m=m'=0,\\ -\epsilon^2\big(\mathcal{R}^{ij}_0\lambda^{\frac14}_{m',i}\psi^1_{m',i},1\big)_{L^2(\Gamma_i)}, & m'>0=m,\\ -\mathbf{i}\epsilon^3\big(\mathcal{R}^{ij}_0 1,\lambda^{\frac14}_{m,j}\psi^1_{m,j}\big)_{L^2(\Gamma_j)}, & m>0=m',\\ -\epsilon^2\big(\mathcal{R}^{ij}_0\lambda^{\frac14}_{m',i}\psi^1_{m',i},\lambda^{\frac14}_{m,j}\psi^1_{m,j}\big)_{L^2(\Gamma_j)}, & m,m'>0.\end{cases}$$

Therefore, $\{c^{ij}_{0m}\}_{m>0}\in\ell^2$, $\{c^{ij}_{m'0}\}_{m'>0}\in\ell^2$, and the operator $\mathcal{A}^{ij}_h$ defined by $\{c^{ij}_{m'm}\}_{m,m'>0}$ is uniformly bounded from ℓ^2 to ℓ^2 with $\|\mathcal{A}^{ij}_h\|=\mathcal{O}(\epsilon^2)$.

Proof. The proof is similar to that of Lemma 2.3. We omit the details here. □

By Theorem 2.1 and Lemma 3.2, Eqs. (3.8) and (3.9) can be transformed to the following two-variable linear equations:

$$\Big\{(e^{\mathbf{i}k}+1)\mathcal{I}_2-(e^{\mathbf{i}k}-1)\mathrm{Diag}\{\Pi_1(\epsilon),\Pi_2(\epsilon)\}+(e^{\mathbf{i}k}-1)\mathbf{i}\epsilon^2\mathcal{M}_2(k)+\mathcal{E}_2\Big\}\begin{bmatrix}b_{0,1}\\ b_{0,2}\end{bmatrix}=0, \tag{3.11}$$

where $\mathcal{I}_N$ denotes the $N\times N$ matrix for $N=2$, the 2×2 matrix $\mathcal{E}_2$ consists of elements $\mathcal{O}(\epsilon^3)$,

$$\mathcal{M}_2(k)=\begin{bmatrix}0 & \dfrac{e^{\mathbf{i}k|C_{12}|}}{2\pi k|C_{12}|}\\ \dfrac{e^{\mathbf{i}k|C_{21}|}}{2\pi k|C_{21}|} & 0\end{bmatrix}, \tag{3.12}$$

$$\Pi_j(\epsilon)=\big[-(\mathcal{S}_{0,j}1,1)_{L^2(\Gamma_j)}+\alpha_j\big]\epsilon\mathbf{i}+\frac{\epsilon^2}{2\pi}. \tag{3.13}$$

By (2.9) and (2.18), the constants $\mathcal{S}_{0,j}$ and $\alpha_j>0$ can be defined similarly but with Γ and the eigenpairs $\{\psi^1_m,\lambda_m\}_{m=0}^\infty$ replaced by Γ_j and $\{\psi^1_{m,j},\lambda_{m,j}\}_{m=0}^\infty$, respectively. We now characterize the resonances of the even modes.

Theorem 3.1. *For $h\ll 1$, the resonance of the even modes of the two-hole slab in $\mathcal{B}$ are*

$$k=k_{m,e}+\mathbf{i}\lambda_j\big(\tilde{\mathcal{M}}_{2,j}(k_{m,e})\big)+\mathbf{i}\lambda_j^2\big(\tilde{\mathcal{M}}_{2,j}(k_{m,e})\big)+\mathcal{O}\big(\epsilon^3_{m,e}\big),\quad j=1,2,\quad m=1,2,\dots,M_e \tag{3.14}$$

for $\epsilon_{m,e}=k_{m,e}h\ll 1$, where $k_{m,e}=(2m-1)\pi$ is a Fabry-Pérot frequency,

$$\begin{aligned}k_{m,e,j}&=k_{m,e}-2\mathbf{i}\Pi_j(\epsilon_{m,e}),\\ \tilde{\mathcal{M}}_{2,j}(k_{m,e})&=-2\mathrm{Diag}\{\Pi_1(k_{m,e,j}h),\Pi_2(k_{m,e,j}h)\}\\ &\quad-2\Pi_j(\epsilon_{m,e})\mathrm{Diag}\{\Pi_1(\epsilon_{m,e}),\Pi_2(\epsilon_{m,e})\}\\ &\quad+2\mathbf{i}\epsilon^2_{m,e}\mathcal{M}_2(k_{m,e}),\end{aligned}$$

$\lambda_j(\tilde{\mathcal{M}}_{2,j}(k_{m,e}))$ indicates the eigenvalue of $\tilde{\mathcal{M}}_{2,j}(k_{m,e})$ closer to $-2\Pi_j(\epsilon_{m,e})$ for $j=1,2$, and the positive integer M_e is the greatest integer such that all the $2M_e$ frequencies lie in $\mathcal{B}$.

Proof. Clearly, (3.11) has a nonzero solution $[b_{0,1},b_{0,2}]^T$ if and only if

$$(e^{\mathbf{i}k}+1)\mathcal{I}_2-(e^{\mathbf{i}k}-1)\mathrm{Diag}\{\Pi_1(\epsilon),\Pi_2(\epsilon)\}+(e^{\mathbf{i}k}-1)\mathbf{i}\epsilon^2\mathcal{M}_2(k)+\mathcal{E}_2 \tag{3.15}$$

has a zero eigenvalue or a zero determinant. Since

$$\left\|(e^{\mathbf{i}k}-1)\mathbf{i}\epsilon^2\mathcal{M}_2+\mathcal{E}_2\right\|_2=\mathcal{O}(\epsilon^2),$$

a resonance k must satisfy

$$(e^{\mathbf{i}k}+1)-(e^{\mathbf{i}k}-1)\Pi_j(\epsilon)=\mathcal{O}(\epsilon^2) \tag{3.16}$$

for some $j=1,2$, otherwise the matrix in (3.15) becomes diagonally dominant. Therefore, $e^{\mathbf{i}k}+1=\mathcal{O}(\epsilon)$. As in Theorem 2.2, $k=k_{m,e}+o(1)$, as $h\to 0$, for some $m=1,2,\ldots,M_e$. Obviously, $\epsilon\approx\epsilon_{m,e}$ so that

$$\delta_{m,e}=k-k_{m,e}\approx(-\mathbf{i})\left[e^{\mathbf{i}(k-k_{m,e})}-1\right]=\mathcal{O}(\epsilon)=\mathcal{O}(\epsilon_{m,e}).$$

Now, (3.16) implies that

$$k=k_{m,e}-2\mathbf{i}\Pi_j(\epsilon_{m,e})+\mathcal{O}(\epsilon_{m,e}^2)=k_{m,e,j}+\mathcal{O}\left(\epsilon_{m,e}^2\right),$$

so that

$$(e^{\mathbf{i}k}+1)\mathcal{I}_2-\tilde{\mathcal{M}}_{2,j}(k_{m,e})+\tilde{\mathcal{E}}_2$$

must have a zero eigenvalue, where it can be seen that the matrix

$$\tilde{\mathcal{E}}_2=\tilde{\mathcal{M}}_{2,j}(k_{m,e})-(e^{\mathbf{i}k}-1)\mathrm{Diag}\{\Pi_1(\epsilon),\Pi_2(\epsilon)\}+(e^{\mathbf{i}k}-1)\mathbf{i}\epsilon^2\mathcal{M}_2(k)+\mathcal{E}_2$$

has elements $\mathcal{O}(\epsilon_{m,e}^3)$. Therefore, it must hold that

$$e^{\mathbf{i}k}+1=\lambda_j\left(\tilde{\mathcal{M}}_{2,j}(k_{m,e})\right)+\mathcal{O}(\epsilon_{m,e}^3),$$

where λ_j denotes the eigenvalue of $\tilde{\mathcal{M}}_{2,j}(k_{m,e})$ closer to $2\Pi_j(\epsilon_{m,e})$. Thus,

$$\delta_{m,e}^2-\mathbf{i}\delta_{m,e}-\lambda_j\left(\tilde{\mathcal{M}}_{2,j}(k_{m,e})\right)+\mathcal{O}\left(\epsilon_{m,e}^3\right)=0,$$

so that

$$\begin{aligned}\delta_{m,e}&=\frac{1}{2}\left(\mathbf{i}-\sqrt{4\lambda_j\left(\tilde{\mathcal{M}}_{2,j}(k_{m,e})\right)-1}\right)+\mathcal{O}\left(\epsilon_{m,e}^3\right)\\&=\mathbf{i}\lambda_j\left(\tilde{\mathcal{M}}_{2,j}(k_{m,e})\right)+\mathbf{i}\lambda_j^2\left(\tilde{\mathcal{M}}_{2,j}(k_{m,e})\right)+\mathcal{O}\left(\epsilon_{m,e}^3\right)\end{aligned}$$

implying (3.14).

We now prove the existence of two solutions. Assume that k lies in the disk

$$D_h=\left\{k\in\mathbb{C}:|k-k_{m,e}|\le h^{\frac{1}{2}}\right\}\subset\mathcal{B}.$$

Then, on the boundary of D_h, all the elements of $(e^{\mathbf{i}k}-1)\mathbf{i}\epsilon^2\mathcal{M}_2(k)+\mathcal{E}_2$ are $\mathcal{O}(h^2)$, so that by linearity

$$|\mathrm{Det}_1-\mathrm{Det}_2|=\mathcal{O}\left(h^{\frac{5}{2}}\right)\le\mathcal{O}(h)=|\mathrm{Det}_2|,$$

where

$$\begin{aligned}
\mathrm{Det}_1&=\left|(e^{\mathbf{i}k}+1)\mathcal{I}_2-(e^{\mathbf{i}k}-1)\mathrm{Diag}\{\Pi_1(\epsilon),\Pi_2(\epsilon)\}-\epsilon(e^{\mathbf{i}k}-1)\mathcal{M}_2(k)-(e^{\mathbf{i}k}-1)\mathcal{E}_2\right|,\\
\mathrm{Det}_2&=\left|(e^{\mathbf{i}k}+1)\mathcal{I}_2-(e^{\mathbf{i}k}-1)\mathrm{Diag}\{\Pi_1(\epsilon),\Pi_2(\epsilon)\}\right|.
\end{aligned}$$

For either $j=1,2$, it is clear that on the boundary of D_h,

$$\left|(e^{\mathbf{i}k}+1)-(e^{\mathbf{i}k}-1)\Pi_j(\epsilon)+\mathbf{i}(k-k_{m,e})\right|=\mathcal{O}(h)\le|-\mathbf{i}(k-k_{m,e})|.$$

The above two inequalities and Rouché's theorem indicate that there are exactly two solutions in D_h. □

The following theorem characterizes the resonances of the odd modes.

Theorem 3.2. *For $h\ll 1$, the resonances of the odd modes of the two-hole slab in $\mathcal{B}$ are*

$$k=k_{m,o}+\mathbf{i}\lambda_j\left(\tilde{\mathcal{M}}_{2,j}(k_{m,o})\right)+\mathbf{i}\lambda_j^2\left(\tilde{\mathcal{M}}_{2,j}(k_{m,o})\right)+\mathcal{O}\left(\epsilon_{m,o}^3\right),\quad j=1,2,\quad m=1,2,\dots,M_o \tag{3.17}$$

for $\epsilon_{m,o}=k_{m,o}h\ll 1$, where $k_{m,o}=2m\pi$ is a Fabry-Pérot frequency,

$$\begin{aligned}
k_{m,o,j}&=k_{m,o}-2\mathbf{i}\Pi_j(\epsilon_{m,o}),\\
\tilde{\mathcal{M}}_{2,j}(k_{m,o})&=-2\mathrm{Diag}\{\Pi_1(k_{m,o,j}h),\Pi_2(k_{m,o,j}h)\}\\
&\quad-2\Pi_j(\epsilon_{m,o})\mathrm{Diag}\{\Pi_1(\epsilon_{m,o}),\Pi_2(\epsilon_{m,o})\}\\
&\quad+2\mathbf{i}\epsilon_{m,o}^2\mathcal{M}_2(k_{m,o}),
\end{aligned}$$

$\lambda_j(\tilde{\mathcal{M}}_{2,j}(k_{m,o}))$ indicates the eigenvalue of $\tilde{\mathcal{M}}_{2,j}(k_{m,o})$ closer to $-2\Pi_j(\epsilon_{m,o})$ for $j=1,2$, and the positive integer M_o is the greatest integer such that all the $2M_o$ frequencies lie in $\mathcal{B}$.

Proof. The proof follows from similar arguments as in Theorem 3.1. □

3.2 Multiple holes

The above results can readily be extended to a slab with the N holes $\{V_{j,h}\}_{j=1}^N$ centered at $\{C_j\}_{j=1}^N$. We state our main result in the following.

Theorem 3.3. *For $h\ll 1$, the resonances of a slab containing $\{V_{j,h}\}_{j=1}^N$ in $\mathcal{B}$ are*

$$k=k_m+\mathbf{i}\lambda_j\big(\tilde{\mathcal{M}}_{N,j}(k_m)\big)+\mathbf{i}\lambda_j^2\big(\tilde{\mathcal{M}}_{N,j}(k_m)\big)+\mathcal{O}\big(\epsilon_m^3\big),\quad j=1,\dots,N,\quad m=1,2,\dots,M \tag{3.18}$$

for $\epsilon_m=k_m h\ll 1$, where $k_m=m\pi$ is a Fabry-Pérot frequency,

$$k_{m,j}=k_m-2\mathbf{i}\Pi_j(\epsilon_m),$$

$$\mathcal{M}_N(k)=\begin{bmatrix} 0 & \dfrac{e^{\mathbf{i}k|C_{12}|}}{2\pi k|C_{12}|} & \cdots & \dfrac{e^{\mathbf{i}k|C_{1N}|}}{2\pi k|C_{1N}|} \\ \dfrac{e^{\mathbf{i}k|C_{21}|}}{2\pi k|C_{21}|} & 0 & \cdots & \dfrac{e^{\mathbf{i}k|C_{2N}|}}{2\pi k|C_{2N}|} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{e^{\mathbf{i}k|C_{N1}|}}{2\pi k|C_{N1}|} & \dfrac{e^{\mathbf{i}k|C_{N2}|}}{2\pi k|C_{N2}|} & \cdots & 0 \end{bmatrix},$$

$$\begin{aligned}\tilde{\mathcal{M}}_{N,j}(k_m)=&-2\mathrm{Diag}\{\Pi_1(k_{m,j}h),\cdots,\Pi_N(k_{m,j}h)\}\\ &-2\Pi_j(\epsilon_m)\mathrm{Diag}\{\Pi_1(\epsilon_m),\cdots,\Pi_N(\epsilon_m)\}+2\mathbf{i}\epsilon_m^2\mathcal{M}_N(k_m),\end{aligned}$$

$|C_{ij}|=|C_i-C_j|$, $\lambda_j(\tilde{\mathcal{M}}_{N,j}(k_m))$ indicates the eigenvalue of $\tilde{\mathcal{M}}_{N,j}(k_m)$ closest to $-2\Pi_j(\epsilon_m)$ for $j=1,\dots,N$, and the positive integer M is the greatest integer such that all the MN frequencies lie in $\mathcal{B}$. Moreover, the imaginary part of each of the MN resonances is $\mathcal{O}(h^2)$.

Proof. The proof of the asymptotics of k is analogous to that of Theorems 3.1 and 3.2. To show that $\mathrm{Im}(k)=\mathcal{O}(h^2)$, we adopt the Gershgorin circle theorem. Each eigenvalue λ_j is estimated by

$$\big|\lambda_j\big(\tilde{\mathcal{M}}_{N,j}(k_m)\big)+2\Pi_j(k_{m,j}h)+2\Pi_j(\epsilon_m)\Pi_j(\epsilon_m)\big|\le\max_i\sum_{n=1,n\neq i}^N\left|2\mathbf{i}\epsilon_m^2\frac{e^{\mathbf{i}k|C_{in}|}}{2\pi k|C_{in}|}\right|,$$

implying that

$$\lambda_j\big(\tilde{\mathcal{M}}_{N,j}(k_m)\big)=-2\Pi_j(\epsilon_m)+\mathcal{O}(h^2).$$

The desired result follows immediately from the definition of Π_j in (3.13). ☐

Remark 3.1. When all holes $\{V_{j,h}\}_{j=1}^N$ are generated by the same Lipschitz domain, say G_1, (3.18) can be simplified to

$$k=k_m-2\mathbf{i}\Pi_1(k_{m,1}h)-2\mathbf{i}\Pi_1^2(\epsilon_m)+4\mathbf{i}\Pi_1^2(k_{m,1}h)-2\epsilon_m^2\lambda_j\big(\mathcal{M}_N(k_m)\big)+\mathcal{O}\big(\epsilon_m^3\big) \tag{3.19}$$

for $\epsilon_m=k_m h\ll 1$, where $k_{m,1}=k_m-2\mathbf{i}\Pi_1(\epsilon_m)$, and $\lambda_j(\mathcal{M}_N(k))$ indicates the j-th eigenvalue (in descending order of real parts) of $\mathcal{M}_N(k)$.

Remark 3.2. Following the same arguments in Section 2.3, it can be shown that for any incident wave field of frequency $k_0=\mathrm{Re}(k)$ with k being some resonances, the Theorem 3.3 implies that the total field can be enhanced by at least a magnitude $\mathcal{O}(h^{-2})$ in each of the N holes for $h\ll 1$.

4 Numerical examples

In this section, we carry out several numerical experiments to validate the derived asymptotic formulae. For simplicity, we shall assume any of the holes $\{V_{j,h}\}_{j=1}^N$ to be either a cuboid or a circular cylinder for that the related eigenpairs $\{\psi_{m,j},\lambda_{m,j}\}_{m=0}^\infty$ for G_j of a circular or rectangular shape are available by the method of separation of variables. For example, for a square $G_1=\{(x,y): -1<x<1,\ -1<y<1\}$,

$$\lambda_{pq}=\frac{1}{4}\pi^2(p^2+q^2),\quad \psi^1_{pq}=\frac{1}{4}\left(e^{\mathbf{i}\frac{p\pi x_1}{2}}+(-1)^p e^{-\mathbf{i}\frac{p\pi x_1}{2}}\right)\left(e^{\mathbf{i}\frac{q\pi x_2}{2}}+(-1)^q e^{-\mathbf{i}\frac{q\pi x_2}{2}}\right),\quad p,q\in\mathbb{N}, \tag{4.1}$$

constitute all eigenpairs. Similarly, for a circle $G_1=\{(x,y):x^2+y^2<1\}$, the eigenpairs are, in polar coordinates,

$$\begin{aligned} &\lambda^e_{pq}=\lambda^o_{pq}=\alpha^2_{pq},\\ &\psi^e_{pq}=C^e_{pq}J_p(\alpha_{pq}r)\cos(p\theta),\\ &\psi^o_{pq}=C^o_{pq}J_p(\alpha_{pq}r)\sin(p\theta),\quad (p,q)\in\mathbb{N}\times\mathbb{Z}^+, \end{aligned} \tag{4.2}$$

where J_p denotes the Bessel function of order p,α_{pq} denotes the q-th smallest root of $J'_p(r)$, and constants C^e_{pq} and C^o_{pq} normalize the eigenfunctions. For more general structures, one has to numerically evaluate these eigenpairs [1,5].

4.1 Nonlinear eigenvalue problem

To clarify the basic idea of our numerical solver, we take the single-hole system (2.15)-(2.16) as an example, the case for multiple holes can be handled similarly. Denote by $\mathcal{L}(k)$ the infinite dimensional linear operator

$$\mathcal{L}(k)=\begin{bmatrix} (e^{\mathbf{i}k}-1)c_{00}-(1+e^{\mathbf{i}k})+1 & c_{10} & \cdots & c_{m'0} & \cdots\\ c_{01}(e^{\mathbf{i}k}-1) & c_{11} & \cdots & c_{m'1} & \cdots\\ \vdots & \vdots & \ddots & \vdots & \ddots\\ c_{0m}(e^{\mathbf{i}k}-1) & c_{1m} & \cdots & c_{m'm} & \cdots\\ \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}-\mathcal{I}, \tag{4.3}$$

where $\mathcal{I}$ denotes the identity operator. The problem can be restated as: Find $k\in\mathcal{B}$ such that there is a nonzero sequence $\{b_0,\{a_{m'}\}\}$ solving

$$\mathcal{L}(k)\begin{bmatrix} b_0\\ \{a_{m'}\}\end{bmatrix}=0. \tag{4.4}$$

We simply truncate $\mathcal{L}$ to an $N\times N$ matrix $\mathcal{L}_N(k)$ by keeping the first N columns and rows of $\mathcal{L}(k)$. The truncation gives rise to the following finite-dimensional nonlinear eigenvalue problem: Find $k\in\mathcal{B}$ that solves

$$\min[\mathrm{eig}\{\mathcal{L}_N(k)\}]=0. \tag{4.5}$$

Here, we adopt the standard Muller's method to compute the resonances k.

To obtain the $N\times N$ matrix $\mathcal{L}_N(k)$, the most time-consuming step is to compute the following integrals:

$$I_{mm'}:=\int_{\Gamma_1}\int_{\Gamma_1}\frac{e^{\mathbf{i}hk|x-y|}}{|x-y|}\psi^1_{m'}(y)dS(y)\overline{\psi^1_m}(x)dS(x),\quad m',m=0,\dots,N-1. \tag{4.6}$$

When kh is small, one can speed up the evaluation of the above integrals based on the following Taylor expansion of $e^{\mathbf{i}hk|x-y|}$:

$$e^{\mathbf{i}hk|x-y|}=1+\mathbf{i}hk|x-y|+\frac{1}{2}(\mathbf{i}hk|x-y|)^2+\frac{1}{6}(\mathbf{i}hk|x-y|)^3+\mathcal{O}\big((hk)^4\big).$$

Then,

$$\begin{aligned}I_{mm'}\approx&\int_{\Gamma_1}\int_{\Gamma_1}\frac{1}{|x-y|}\psi^1_{m'}(y)dS(y)\overline{\psi^1_m}(x)dS(x)+\mathbf{i}hk\delta_{m0}\delta_{m'0}\\&-\frac{h^2k^2}{2}\int_{\Gamma_1}\int_{\Gamma_1}|x-y|\psi^1_{m'}(y)dS(y)\overline{\psi^1_m}(x)dS(x)\\&-\frac{\mathbf{i}h^3k^3}{6}\int_{\Gamma_1}\int_{\Gamma_1}|x-y|^2\psi^1_{m'}(y)dS(y)\overline{\psi^1_m}(x)dS(x).\end{aligned}$$

Note that the last term in fact can be separated as the product of two surface integrals for that $|x-y|^2=|x|^2-2x\cdot y+|y|^2$. Thus, we only need to deal with the first two quadruple integrals, which are independent of k. All the integrals in $I_{mm'}$ are evaluated directly by two built-in functions `integral` and `quad2d` of MATLAB.

For comparison, one similarly evaluates α and $(S_0 1,1)_{L^2(\Gamma_1)}$ in (2.23) such that the asymptotic results in Theorems 2.2 and 2.4 provide approximate frequencies k.

4.2 Numerical results

In the first example, there is only a single square hole centered at the origin that

$$\Gamma_h=\big\{x\in\mathbb{R}^3:|x_1|<h,|x_2|<h,x_3=0\big\}.$$

We compare the asymptotic results and the numerical results near π and 2π, while the value of h varies from 0.2 to 0.0125. The results are shown in Tables 1 and 2, where k_{num} denotes the numerical value of k and the fourth column lists absolute errors between k_{num} and the asymptotic formula. In the two tables, we see clearly that when h is halved, the absolute errors for formulae (2.21) and (2.35) appear to be roughly 1/8 of the previous one, agreeing with the indicated error $\mathcal{O}(h^3)$ in Theorems 2.2 and 2.4.

Table 1: The resonance of a square hole near π.

h	k_{num}	Eq. (2.21)	Err($\mathcal{O}(h^3)$)
0.2	2.3486071−0.1834275**i**	2.3322908−0.1348435**i**	5.1E-2
0.1	2.6697389−0.0728046**i**	2.6669942−0.0796873**i**	7.4E-3
0.05	2.8810272−0.0237653**i**	2.8802092−0.0256689**i**	2.1E-3
0.025	3.0045698−0.0068319**i**	3.0044676−0.0071356**i**	3.2E-4
0.0125	3.0713782−0.0018320**i**	3.0713960−0.0018737**i**	4.5E-5

Table 2: The resonance of a square hole near 2π.

h	k_{num}	Eq. (2.35)	Err($\mathcal{O}(h^3)$)
0.2	4.8970464−0.6165919**i**	4.1820329−0.5393740**i**	7.2E-1
0.1	5.4096894−0.2612578**i**	5.3038292−0.3187492**i**	1.2E-1
0.05	5.7779487−0.0907690**i**	5.7585335−0.1026754**i**	2.3E-2
0.025	6.0118333−0.0269155**i**	6.0088173−0.0285424**i**	3.4E-3
0.0125	6.1431457−0.0072962**i**	6.1427846−0.0074948**i**	4.1E-4

In the second example, we assume that there is only a single circular hole centered at the origin that

$$\Gamma_h = \left\{ x \in \mathbb{R}^3 : \left(x_1^2 + x_2^2\right)^{\frac{1}{2}} < h, x_3 = 0 \right\}.$$

Same as the first example, we compare the asymptotic results and the numerical results near π and 2π, while the value of h varies from 0.0125 to 0.2. The results are shown in Tables 3 and 4, where k_{num} denotes the numerical value of k and the fourth column lists absolute errors between k_{num} and the asymptotic formula.

Table 3: The resonance of a circular hole near π.

h	k_{num}	Eq. (2.21)	Err($\mathcal{O}(h^3)$)
0.2	2.3995389−0.1556778**i**	2.3957256−0.1330007**i**	2.3E-2
0.1	2.7056976−0.0600111**i**	2.7040269−0.0659731**i**	6.2E-3
0.05	2.9031657−0.0191609**i**	2.9025822−0.0205836**i**	1.5E-3
0.025	3.0168650−0.0054375**i**	3.0167762−0.0056572**i**	2.4E-4
0.0125	3.0778398−0.0014483**i**	3.0778407−0.0014782**i**	3.0E-5

Table 4: The resonance of a circular hole near 2π.

h	k_{num}	Eq. (2.35)	Err($\mathcal{O}(h^3)$)
0.2	4.9715497−0.5215566**i**	4.4937909−0.5320028**i**	4.8E-1
0.1	5.4667041−0.2173300**i**	5.3894501−0.2638924**i**	9.0E-2
0.05	5.8181787−0.0736574**i**	5.8040016−0.0823345**i**	1.7E-2
0.025	6.0356732−0.0214768**i**	6.0334796−0.0226288**i**	2.5E-3
0.0125	6.1559559−0.0057729**i**	6.1556768−0.0059129**i**	3.1E-4

In the third example, a slab with a square hole and a circular hole is considered. Specifically,

$$\Gamma_{1,h}=\left\{x\in\mathbb{R}^3: |x_1|<h, |x_2|<h, x_3=0\right\},$$
$$\Gamma_{2,h}=\left\{x\in\mathbb{R}^3: \left((x_1-1)^2+x_2^2\right)^{\frac{1}{2}}<h, x_3=0\right\}.$$

We compare the asymptotic results and the numerical results near π, where two resonances are encountered, while h varies from 0.0125 to 0.2. The results are shown in Tables 5 and 6. We see clearly that when h is halved, the absolute errors appear to be $1/8$ of the previous one in the first table and the same accuracy is achieved after h is reduced to less than 0.05 in the second table. In all tables, it can be seen that the imaginary part of a resonance is reduced to about $1/4$ of itself when h is halved indicating that the resonances have $\mathcal{O}(h^2)$ imaginary parts for $h\ll 1$.

Table 5: The resonance of a square hole and a circular hole near Π_1.

h	k_{num}	Eq. (3.14)	Err($\mathcal{O}(h^3)$)
0.2	2.3354764−0.1061379**i**	2.2290167−0.1167617**i**	1.1E-1
0.1	2.6624839−0.0587463**i**	2.6483666−0.0751923**i**	2.2E-2
0.05	2.8792254−0.0222390**i**	2.8774397−0.0251300**i**	3.4E-3
0.025	3.0042659−0.0067074**i**	3.0041014−0.0070932**i**	4.2E-4
0.0125	3.0713358−0.0018233**i**	3.0713499−0.0018709**i**	5.0E-5

Table 6: The resonance of a square hole and a circular hole near Π_2.

h	k_{num}	Eq. (3.14)	Err($\mathcal{O}(h^3)$)
0.2	2.4223191−0.2487429**i**	2.2074658−0.1708974**i**	2.3E-1
0.1	2.7144004−0.0739789**i**	2.6398102−0.0821431**i**	7.5E-2
0.05	2.9049845−0.0206113**i**	2.8750643−0.0260130**i**	3.0E-2
0.025	3.0171292−0.0055545**i**	3.0171407−0.0056996**i**	1.5E-4
0.0125	3.0778591−0.0014565**i**	3.0778867−0.0014810**i**	3.7E-5

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