Commun. Math. Res.
doi: 10.4208/cmr.2023-0034

Vol. **x**, No. x, pp. 1-37
x 2024

Symplectic Conditions on Grassmannian, Flag, and Schubert Varieties

Jiajun Xu and Guanglian Zhang*

School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China.

Received 1 August 2023; Accepted 3 January 2024

Abstract. In this paper, a description of the set-theoretical defining equations of symplectic (type C) Grassmannian/flag/Schubert varieties in corresponding (type A) algebraic varieties is given as linear polynomials in Plücker coordinates, and it is proved that such equations generate the defining ideal of variety of type C in those of type A. As applications of this result, the number of local equations required to obtain the Schubert variety of type C from the Schubert variety of type A is computed, and further geometric properties of the Schubert variety of type C are given in the aspect of complete intersections. Finally, the smoothness of Schubert variety in the non-minuscule or cominuscule Grassmannian of type C is discussed, filling gaps in the study of algebraic varieties of the same type.

AMS subject classifications: 14M15, 14L30, 15A15

Key words: Grassmannian variety, generalized flag variety, Schubert variety, Plücker embedding, complete intersection.

---

1 Introduction

Grassmannian and flag varieties, which stem from linear algebra, are important study objects in the interplay of algebraic geometry, representation theory, and combinatorics. The symplectic Grassmannian and flag variety have also attracted considerable interest from researchers (e.g. [1,7]). As one of the best-understood examples of singular projective varieties, the Schubert variety plays an important

*Corresponding author. *Email addresses:* `g.l.zhang@sjtu.edu.cn` (G. Zhang), `s1gh1995@sjtu.edu.cn` (J. Xu)

role in the study of generalized Grassmannian/flag varieties. Its relation with the cohomology theory on Grassmannian was first proposed by Hermann Schubert as early as the 19th century and later featured as the 15th problem among Hilbert's famous 23 problems.

Let k be an algebraically closed field with $\mathrm{char}(k)=0$, and $e_1,e_2,\cdots,e_n$ the standard basis of the linear space k^n. For any $d\leq n$, put

$$\begin{aligned} I_{d,n} &= \left\{\underline{i}=(i_1,i_2,\cdots,i_d)\,|\,1\leq i_1<i_2<\cdots<i_d\leq n\right\} \\ &= \left\{d-\text{subsets of } \{1,2,\ldots,n\}\right\}. \end{aligned}$$

Then $\{e_{\underline{i}}=e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_d}\,|\,\underline{i}\in I_{d,n}\}$ forms a basis of $\wedge^d k^n$. Denote its dual basis in $(\wedge^d k^n)^*$ by $\{p_{\underline{i}}\,|\,\underline{i}\in I_{d,n}\}$

$$p_{\underline{i}}(e_{\underline{j}})=\begin{cases} 1, & \text{if} \quad \underline{i}=\underline{j}, \\ 0, & \text{if} \quad \underline{i}\neq\underline{j}. \end{cases}$$

Then, $\{p_{\underline{i}}\,|\,\underline{i}\in I_{d,n}\}$ can be viewed as the homogeneous (projective) coordinates on $\mathbb{P}(\wedge^d k^n)$. Let $\mathrm{Gr}(d,n)$ be the Grassmannian variety formed by the d-dimensional subspace of k^n (if in the scheme-theoretical language, the closed points only). There is the famous Plücker embedding

$$\begin{aligned} \mathrm{Gr}(d,n) &\to \mathbb{P}(\wedge^d k^n)=\mathbb{P}^{\binom{n}{d}-1}, \\ \mathrm{Span}\{\boldsymbol{v_1},\boldsymbol{v_2},\cdots,\boldsymbol{v_d}\} &\mapsto [\boldsymbol{v_1}\wedge\boldsymbol{v_2}\wedge\cdots\wedge\boldsymbol{v_d}]. \end{aligned}$$

Additionally, $\{p_{\underline{i}}\,|\,\underline{i}\in I_{d,n}\}$ can be regarded as the homogeneous coordinates on $\mathrm{Gr}(d,2n)$. Everything discussed in this article is under this Plücker embedding.

Let

$$\boldsymbol{J}=\begin{bmatrix} & & 0 & & 1 \\ & & & \cdots & \\ & & 1 & & 0 \\ 0 & & -1 & & \\ & \cdots & & & \\ -1 & & 0 & & \end{bmatrix}_{2n\times 2n}.$$

Then, the symplectic Grassmannian is exactly

$$\mathrm{Gr}^C(d,2n)=\left\{V\in\mathrm{Gr}(d,2n)\,|\,V\perp \boldsymbol{J}V\right\},\quad 1\leq d\leq n,$$

where, for the column vectors $\boldsymbol{u},\boldsymbol{v}\in k^{2n}$, $\boldsymbol{u}\perp\boldsymbol{v}$ means $\boldsymbol{u}^T\boldsymbol{v}=0$. We employ the superscript C because of its connection to the classic linear algebraic group of type C, i.e., the symplectic group Sp_{2n}. In contrast, we sometimes use a superscript A for objects corresponding to the special linear group SL_{2n}.

Similarly, we can define the (partial) flag variety and symplectic flag variety

$$\begin{aligned}&\mathrm{Fl}_{2n}(1,2,\dots,n)=\{0\subset V_1\subset V_2\subset\cdots\subset V_n\subset k^{2n}\mid \dim V_t=t\},\\&\mathrm{Fl}^{C}_{2n}(1,2,\dots,n)=\{0\subset V_1\subset V_2\subset\cdots\subset V_n\subset k^{2n}\mid \dim V_t=t,V_n\perp JV_n\}.\end{aligned}$$

In $\mathrm{Gr}(d,2n)$ (resp. $\mathrm{Fl}_{2n}(1,2,\dots,n)$), $\mathrm{Gr}^C(d,2n)$ (resp. $\mathrm{Fl}^C_{2n}(1,2,\dots,n)$) is a closed subvariety. We can identify them with the homogeneous spaces of SL_{2n} and Sp_{2n}, respectively. Let us fix the upper triangular Borel subgroups of SL_{2n} and Sp_{2n}. Then, by taking the closures (under Zariski topology) of orbits of Borel action, we can obtain the Schubert varieties of types A and C, see Section 3 for more details.

For the comparative study between Schubert varieties of types A and C, especially on the geometric aspects such as smoothness and singular locus, one widely used method is viewing Sp_{2n} as a fixed point set of a certain involution on SL_{2n} (cf. [10, 11, 14], [2, p.29]). And another technique, which is equally straightforward but has received less attention, is to use the defining ideals of varieties. The defining ideal is formed by objects in the homogeneous coordinate ring of Schubert variety of type A that are identically zero on Schubert variety of type C. In contrast to discussing the required conditions in the set-theoretical meaning, the discussion of obtaining variety of type C from the intersection of variety of type A and giving the defining ideal is complicated. It is also a classically important problem to prove that some natural set-theoretical conditions generate the defining ideal: one example is on the determinantal variety formed by $m\times n$ matrices of rank not greater than t, where $n,m,t\in\mathbb{N}^+$. Clearly, the rank of a matrix is not greater than t if and only all of its $t+1$-minors are zero. However, to explain why the defining ideal of such a determinantal variety is generated by $t+1$-minor functions (or equivalently these functions generate a radical ideal), we need a relatively long discussion [2, p.180]. In our issue, the type C Grassmannian has been recognized as a linear slice of type A Grassmannian in many elementary research before, i.e. one has known that the defining ideal of type C Grassmannian is generated by linear polynomials [5]. But a set of generators, which are naturally interpreted and easy to apply, and the analysis on it are still scarce.

So in our work, the problems are to describe the symplectic orthogonal condition "$V\perp JV$" via some proper formulas and to determine whether these conditions generate the defining ideal of Grassmannian/flag/Schubert varieties of type C in the corresponding varieties of type A. An existing relevant work on this topic is from De Concini [4]. In his study of the basis for coordinate rings of symplectic determinantal variety, he proposed an algorithm that indicated that the defining ideal could be generated by a series of special linear equations [4, (1.8)] (but still not symplectic orthogonal conditions, at least a good interpretation is lacking). We refer to this result in further depth in Section 3 (see also [6] and Lakshmibai's work on admissible pair [13] for more comments, which also reveals that De Concini's equations were considered to be a most advanced existing result for a long time in the past). Although De Concini has not developed a better discussion on those equations because of his focus on the basis for the coordinate ring, he did spark interest in various problems: the (local) orthogonal relation appears to be a quadratic equation rather than his linear one intuitively. While we use a different setting of equations than did De Concini, this problem will be clarified in the next section. With a natural interpretation, we obtain the expression of (set-theoretical) defining equations of Grassmannian/flag/Schubert variety of type C in Grassmannian/flag/Schubert variety of type A as linear polynomials in the global homogeneous coordinates $p_{\underline{i}}$. Then, in the Grassmannian, we show that the restrictions of these equations generate the (scheme-theoretical) defining ideal of Schubert variety of type C in the varieties of type A. For flag varieties, a similar analysis is also conducted.

We should note that the equations in Section 2 also appeared in the cryptography research paper from Carrillo-Pacheco and Zaldivar [16]. The main result of their work is to prove that the equations set-theoretically define the Lagrangian Grassmannian $\mathrm{Gr}^C(n,2n)$ in $\mathrm{Gr}(n,2n)$. In contrast, we obtain the equations in different methods from theirs and discuss on a more generalized scope rather than only Lagrangian Grassmannian. Further, the most important novelties of ours in this part are the natural interpretation to the equations and the scheme-theoretical conclusions in Section 3.

After determining the defining ideal, the restriction of some of our equations on the Schubert varieties are zero. Then, counting the number of nonzero equations has its own importance: it provides a tool for further comparative studies between the Schubert varieties of types A and C. For an algebraic variety X and its subvariety Y, we say Y is a complete intersection of X if the defining ideal of Y in X is generated by $codim_X Y$ equations. Moreover, Y is called a local complete intersection of X if there is an open cover $X=\cup U_i$ such that $Y\cap U_i$ is a complete intersection of $X\cap U_i$ for every i. Especially, at this time, if X is smooth, then it is known that Y is Gorenstein and Cohen–Macaulay, which implies that Y has mild singularities. Being a local complete intersection of one smooth variety is an intrinsic property, so at this time, we can drop the emphasis on X. By counting the nonzero generators of the defining ideal, we can give the conditions for Schubert varieties of type C to become a complete intersection of variety of type A (or intrinsically).

Finally, we discuss a research gap related to the smoothness of Schubert varieties in $\mathrm{Gr}^C(d,2n), 1<d<n$. Existing studies on the symplectic Grassmannian concentrate on $\mathrm{Gr}^C(n,2n)$ and trivial $\mathrm{Gr}^C(1,2n)$ because their corresponding maxi-

mal parabolic subgroups in Sp_{2n} are the only cominuscule or minuscule ones [15]. However, every maximal parabolic subgroup of SL_{2n} is cominuscule and minuscule, and the research on $\mathrm{Gr}(d,2n)$ is fairly mature for any $1 \le d \le 2n$. Thus, the defining ideal helps us to establish the connection between the Schubert varieties in $\mathrm{Gr}(d,2n)$ and to those in $\mathrm{Gr}^C(d,2n), 1<d<n$. On $\mathrm{Gr}^C(n,2n)$, there is an interesting result that the symplectic Schubert variety must be smooth if the corresponding Schubert variety of type A is already smooth [15]. A similar argument holds for symplectic Schubert varieties in flag variety [14]. However, we note that this does not always hold for Schubert varieties in $\mathrm{Gr}^C(d,2n), 1<d<n$.

In addition to geometric research, the results of this paper are also expected to be applied to representation theory problems in the future. For example, the closure of the isomorphic classes of quiver representation variety has always shown a profound connection with Schubert and Kazhdan-Lusztig varieties (cf. [3,8,9]), and this connection has gradually been revealed more clearly in recent years, with research methods that heavily rely on accurate calculations of defining ideals. We will also continue to explore research possibilities in this area in the future.

The rest of this paper is organized as follows. In Section 2, we collect preliminaries on Grassmannian variety, then provide a sketch of the equations we require. As a special result, the local relations, Proposition 2.3, are computed. In Section 3, we consider flag varieties and Schubert varieties. We introduce the work of De Concini, which enables us to focus on linear equations rather than higher degree ones. Then, employing an inductive method that might also be useful for further research, the main Theorems 3.4 and 3.5 are proved. The generators of the defining ideal of Schubert varieties of type C in the varieties of type A are identified. Moreover, utilizing the local relations provided in Section 2, we obtain the local defining ideals. In Section 4, the number of nonzero local equations on specific Schubert varieties is considered. Proposition 4.1 reveals that in some special affine open subset, the number is exactly equal to the codimension, but in general, this finding does not hold. The relation among this number in different affine open subsets is discussed in Lemma 4.3 and its Corollary 4.1. As the main result of this section, we find the precise number of nonzero local equations in Proposition 4.4, and the conditions that are required for the Schubert variety of type C to be a local complete intersection in the variety of type A are derived in Theorem 4.1. In Section 5, the symplectic conditions on tangent spaces are discussed. We compute the codimension of tangent spaces, and a result on the smoothness of Schubert variety in $\mathrm{Gr}^C(d,2n), 1<d<n$, is noted. This result shows a difference between Schubert varieties in $\mathrm{Gr}^C(d,2n), 1<d<n$ and the ones in Sp_{2n}/Q, where Q is a Borel subgroup or minuscule/cominuscule maximal parabolic subgroup.

2 Symplectic Grassmannian and flag varieties

2.1 Grassmannian and Plücker coordinates

For all d,n,t and $\underline{i}\in I_{d,n}$, we denote the t-th element in ascending order in $\underline{i}$ by i_t.

If $V\in\mathrm{Gr}(d,n)$ has basis $\boldsymbol{v_1},\boldsymbol{v_2},\cdots,\boldsymbol{v_d}$ (as column vectors), then we can simultaneously determine all the projective coordinates

$$p_{\underline{i}}(V)=\det(\boldsymbol{M}_{i_1i_2\cdots i_d}),$$

where $\boldsymbol{M}$ is the matrix $[\boldsymbol{v_1},\boldsymbol{v_2},\cdots,\boldsymbol{v_d}]$ and $\boldsymbol{M}_{i_1i_2\cdots i_d}$ takes the $i_1,i_2,\cdots,i_d$-th rows of $\boldsymbol{M}$. As projective coordinates, their quotient $p_{\underline{i}}/p_{\underline{j}}$ remains invariant if we choose a different basis of V(with $p_{\underline{j}}(V)\neq 0$).

Note that if $p_{\underline{j}}(V)\neq 0$, then there must be a suitable basis of V that makes $\boldsymbol{M}_{j_1j_2\cdots j_d}=\boldsymbol{Id}_{d\times d}$. We call such $\boldsymbol{M}$ $\underline{j}$-standard. Additionally, if $\boldsymbol{M}$ is $\underline{j}$-standard, we can directly compute

$$\frac{p_{\underline{i}}}{p_{\underline{j}}}(V)=\det(\boldsymbol{M}_{i_1i_2\cdots i_d}).$$

Let

$$\boldsymbol{J}=\begin{bmatrix} & & & & 0 & & 1\\ & & & & & \cdots & \\ & & & & 1 & & 0\\ 0 & & & -1 & & & \\ & \cdots & & & & & \\ -1 & & 0 & & & & \end{bmatrix}_{2n\times 2n}.$$

The symplectic group $\mathrm{Sp}_{2n}=\{\boldsymbol{M}\in\mathrm{SL}_{2n}\mid \boldsymbol{M}^T\boldsymbol{J}\boldsymbol{M}=\boldsymbol{J}\}$. We fix the group of upper triangular matrices B^A (resp. B^C) in SL_{2n} (resp. in Sp_{2n}) as the standard Borel subgroup and the group of diagonal matrices as the maximal torus in B^A (resp. in B^C). Take $\epsilon_t(\mathbf{diag}(a_1,a_2,\cdots,a_{2n}))=a_t$ ($1\leq t\leq 2n$ for SL_{2n} and $1\leq t\leq n$ for Sp_{2n}) to be the canonical basis for multiplicative characters of the maximal torus. With respect to this maximal torus and standard Borel, we have the simple roots $S(\mathrm{SL}_{2n})=\{\epsilon_t-\epsilon_{t+1}\mid 1\leq t\leq 2n-1\}$ of SL_{2n}, and $S(\mathrm{Sp}_{2n})=\{\epsilon_t-\epsilon_{t+1}\mid 1\leq t\leq n-1\}\cup\{2\epsilon_n\}$ of Sp_{2n}.

Associated with every subset of simple roots S', a unique parabolic subgroup containing the standard Borel can be determined [18, p.147]. For $1\leq d\leq n$, let P_d^A be the parabolic subgroup in SL_{2n} associated with $S(\mathrm{SL}_{2n})\backslash\{\epsilon_d-\epsilon_{d+1}\}$ (then P_d^A is the group of "$(d,n-d)$-blocked" upper triangular matrices), and let P_d^C be the parabolic subgroup in Sp_{2n} associated with the set

$$\begin{cases} S(\mathrm{Sp}_{2n})\backslash\{\epsilon_d-\epsilon_{d+1}\}, & \text{if} \quad 1\leq d\leq n-1, \\ S(\mathrm{Sp}_{2n})\backslash\{2\epsilon_n\}, & \text{if} \quad d=n. \end{cases}$$

Note that $P_d^C=P_d^A\cap\mathrm{Sp}_{2n}$, then, we have the induced embedding $\mathrm{Sp}_{2n}/P_d^C\to\mathrm{SL}_{2n}/P_d^A$. We have the quotients $\mathrm{SL}_{2n}/P_d^A\cong\mathrm{Gr}(d,2n)$ as homogeneous SL_{2n}-space [12, p.168]. Under this identification,

$$\mathrm{Sp}_{2n}/P_d^C\cong\mathrm{Gr}^C(d,2n)=\{U\in\mathrm{Gr}(d,2n)\,|\,V\perp \boldsymbol{J}V\}.$$

We can quickly determine the following dimension formula from the root data.

Proposition 2.1. $\dim\mathrm{Gr}^C(d,2n)=\dim\mathrm{Gr}(d,2n)-\binom{d}{2},\forall 1\leq d\leq n.$

2.2 The symplectic conditions and their local relations

For any sequence of integers $a_1,a_2,\cdots,a_l$, let $\tau(a_1,a_2,\cdots,a_l)=\#\{1\leq s<t\leq l\,|\,a_s>a_t\}$ be the inversion number. Our first goal is to show that

Theorem 2.1. *$\mathrm{Gr}^C(d,2n)$ is the (set-theoretical) intersection of $\mathrm{Gr}(d,2n)$ with hyperplanes*

$$E_{\underline{i'}}=\sum_{t=1}^{n}(-1)^{\tau(i'_1,i'_2,\cdots,i'_{d-2},t,2n+1-t)}p_{\underline{i'}\cup\{t,2n+1-t\}},\quad \underline{i'}\in I_{d-2,2n}.$$

Note that $\underline{i'}$ can be viewed as a subset of $\{1,2,\ldots,2n\}$, so the notation $\underline{i'}\cup\{t,2n+1-t\}$ makes sense. If the subscript of the Plücker coordinate has overlap, i.e., $\#(\underline{i'}\cup\{t,2n+1,t\})<d$, we set this term (also the inversion number) to zero.

In $\mathbb{P}(\wedge^d k^{2n})$ with homogeneous coordinates $\{p_{\underline{i}}\,|\,\underline{i}\in I_{d,2n}\}$, the canonical affine open cover of $\mathrm{Gr}(d,2n)$ is formed by

$$A_{\underline{i}}=\big\{V\in\mathrm{Gr}(d,2n)\,|\,p_{\underline{i}}(V)\neq 0\big\}\cong\mathbb{A}^{d(2n-d)},\quad \underline{i}\in I_{d,n}.$$

Now, we consider the affine open subset $A_{\underline{i}}$ and the local relations of hyperplanes mentioned above.

For a matrix $\boldsymbol{M}_{n\times d}$, denote its columns by $\boldsymbol{c_1},\boldsymbol{c_2},\cdots,\boldsymbol{c_d}$, and set

$$C(\boldsymbol{M},s,t)=\boldsymbol{c_s}^T\boldsymbol{J}\boldsymbol{c_t}.$$

Clearly, a vector space $V\in\mathrm{Gr}(d,2n)$ belongs to $\mathrm{Gr}^C(d,2n)$ if for all matrix presentation $\boldsymbol{M},\forall s\neq t,C(\boldsymbol{M},s,t)=0$. Moreover, note that $C(\boldsymbol{M},s,s)=0,C(\boldsymbol{M},s,t)=-C(\boldsymbol{M},t,s)$ for all s,t. But locally, for example, in A_{id}, $id=(1,2,\ldots,d)\in I_{d,2n}$, the equations $C(\boldsymbol{M},s,t)$ do not take linear forms in local coordinates.

Example 2.1. $d=3,n=4$, (without confusion we omit the comma ',' in $\underline{i}$)

$$\boldsymbol{M}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ x_{41} & x_{42} & x_{43} \\ x_{51} & x_{52} & x_{53} \\ x_{61} & x_{62} & x_{63} \\ x_{71} & x_{72} & x_{73} \\ x_{81} & x_{82} & x_{83} \end{bmatrix}$$

is id-standard. $C(\boldsymbol{M},1,2)=x_{82}+x_{41}x_{52}-x_{71}-x_{42}x_{51}$ is of degree 2 in x_{ij}. Meanwhile, we can write this equation as

$$C(\boldsymbol{M},1,2)=-\frac{p_{138}}{p_{123}}+\frac{p_{345}}{p_{123}}-\frac{p_{237}}{p_{123}}=\frac{E_3}{p_{123}}.$$

Or equivalently,

$$x_{41}x_{52}-x_{42}x_{51}=-\frac{p_{234}p_{135}}{p_{123}^2}-\frac{-p_{134}p_{235}}{p_{123}^2}=\frac{p_{345}p_{123}}{p_{123}^2}.$$

This nonhomogeneous equation provides a common factor p_{123} in the form of global coordinates.

Proposition 2.2. *If $V\in A_{\underline{i}}$ and $\boldsymbol{M}$ is an $\underline{i}$-standard matrix presentation of V, then $\forall s<t$, we have*

$$C(\boldsymbol{M},s,t)=(-1)^{s+t}\frac{E_{\underline{i'}}}{p_{\underline{i}}}(V),$$

where $\underline{i'}=\underline{i}\backslash\{i_s,i_t\}\in I_{d-2,2n}$, and

$$E_{\underline{i'}}=\sum_{t=1}^{n}(-1)^{\tau(i'_1,i'_2,\cdots,i'_{d-2},t,2n+1-t)}p_{\underline{i'}\cup\{t,2n+1-t\}},\quad \underline{i'}\in I_{d-2,2n}.$$

Proof. Note that $C(\boldsymbol{M},s,t)=-C(\boldsymbol{M},t,s)$, so in this proposition, we discuss only the case $s<t$. Then, $\tau(\underline{i'},i_s,i_t)=2d-(s+t)$. Denote the entries of $\boldsymbol{M}$ by x_{ij}, and set

$$C_k(\boldsymbol{M},s,t)=x_{ks}x_{2n+1-k,t}-x_{2n+1-k,s}x_{kt}\quad \text{for}\quad 1\leq k\leq n.$$

Then,

$$C(\boldsymbol{M},s,t)=\sum_{k=1}^{n}C_k(\boldsymbol{M},s,t).$$

Now, consider $C_k(\boldsymbol{M},s,t)$.

Case 1. $k \notin \underline{i}$ and $2n+1-k \notin \underline{i}$, i.e. $\{k,2n+1-k\} \cap \underline{i} = \emptyset$. Then,

$$\begin{aligned} C_k(\boldsymbol{M},s,t) &= x_{ks}x_{2n+1-k,t} - x_{2n+1-k,s}x_{kt} \\ &= (-1)^{\tau(\underline{i'},i_s,i_t)+\tau(\underline{i'},k,2n+1-k)} \frac{p_{\{k,2n+1-k\}\cup \underline{i'}}}{p_{\underline{i}}}(V). \end{aligned}$$

Case 2. $\{k,2n+1-k\} \cap \underline{i} \neq \emptyset$, but $\{k,2n+1-k\} \cap \{i_s,i_t\} = \emptyset$. Then,

$$C_k(\boldsymbol{M},s,t) = 0 = (-1)^{\tau(\underline{i'},i_s,i_t)+\tau(\underline{i'},k,2n+1-k)} \frac{p_{\{k,2n+1-k\}\cup \underline{i'}}}{p_{\underline{i}}}(V),$$

because $\{k,2n+1-k\} \cap \underline{i'} \neq \emptyset$.

Case 3. $\{k,2n+1-k\} \cap \{i_s,i_t\} \neq \emptyset$.

If $k = i_s$, then no matter whether $2n+1-i_s = i_t$ or not,

$$C_k(\boldsymbol{M},s,t) = x_{2n+1-i_s,t} = (-1)^{\tau(\underline{i'},i_s,i_t)+\tau(\underline{i'},i_s,2n+1-i_s)} \frac{p_{\{i_s,2n+1-i_s\}\cup \underline{i'}}}{p_{\underline{i}}}(V).$$

If $k = i_t$, then

$$\begin{aligned} C_k(\boldsymbol{M},s,t) = -x_{2n+1-i_t,s} &= (-1)^{\tau(\underline{i'},i_s,i_t)+\tau(\underline{i'},2n+1-i_t,i_t)+1} \frac{p_{\{i_t,2n+1-i_t\}\cup \underline{i'}}}{p_{\underline{i}}}(V) \\ &= (-1)^{\tau(\underline{i'},i_s,i_t)+\tau(\underline{i'},i_t,2n+1-i_t)} \frac{p_{\{i_t,2n+1-i_t\}\cup \underline{i'}}}{p_{\underline{i}}}(V). \end{aligned}$$

If $2n+1-k = i_s$, then

$$\begin{aligned} C_k(\boldsymbol{M},s,t) = -x_{2n+1-i_s,t} &= (-1)^{\tau(\underline{i'},i_s,i_t)+\tau(\underline{i'},2n+1-i_s,i_s)+1} \frac{p_{\{i_s,2n+1-i_s\}\cup \underline{i'}}}{p_{\underline{i}}}(V) \\ &= (-1)^{\tau(\underline{i'},i_s,i_t)+\tau(\underline{i'},2n+1-i_s,i_s)} \frac{p_{\{i_s,2n+1-i_s\}\cup \underline{i'}}}{p_{\underline{i}}}(V). \end{aligned}$$

If $2n+1-k = i_t$, then

$$C_k(\boldsymbol{M},s,t) = x_{2n+1-i_t,s} = (-1)^{\tau(\underline{i'},i_s,i_t)+\tau(\underline{i'},2n+1-i_t,i_t)} \frac{p_{\{i_t,2n+1-i_t\}\cup \underline{i'}}}{p_{\underline{i}}}(V).$$

Thus, for $\forall 1 \le k \le n$,

$$\begin{aligned} C_k(\boldsymbol{M},s,t) &= (-1)^{\tau(\underline{i'},i_s,i_t)+\tau(\underline{i'},k,2n+1-k)} \frac{p_{\{k,2n+1-k\}\cup \underline{i'}}}{p_{\underline{i}}}(V), \\ C(\boldsymbol{M},s,t) &= \sum_{k=1}^{n} C_k(\boldsymbol{M},s,t) = (-1)^{\tau(\underline{i'},i_s,i_t)} \frac{E_{\underline{i'}}}{p_{\underline{i}}}(V) = (-1)^{s+t} \frac{E_{\underline{i'}}}{p_{\underline{i}}}(V). \end{aligned}$$

The proof is complete. $\square$

Since $A_{\underline{i}}, \underline{i}\in I_{d,2n}$ forms an open cover of $\mathrm{Gr}(d,2n)$, we obtain

Theorem 2.2. *$\mathrm{Gr}^C(d,2n)$ is the (set-theoretical) intersection of $\mathrm{Gr}(d,2n)$ with hyperplanes*

$$E_{\underline{i'}}=\sum_{t=1}^{n}(-1)^{\tau(i'_1,i'_2,\cdots,i'_{d-2},t,2n+1-t)}p_{\underline{i'}\cup\{t,2n+1-t\}}, \quad \underline{i'}\in I_{d-2,2n}.$$

Proposition 2.3. *$\forall \underline{j'}\in I_{d-2,2n}$ and $\underline{i}\in I_{d,2n}$,*

$$E_{\underline{j'}}=\sum_{l=1}^{d}\sum_{k=1,k<l}^{d}\frac{E_{\underline{i}\backslash\{i_k,i_l\}}}{p_{\underline{i}}}\cdot(-1)^{k+l+\tau(\underline{j'},i_k,i_l)}p_{\underline{j'}\cup\{i_k,i_l\}}$$

on $A_{\underline{i}}$, the affine open subset of $\mathrm{Gr}(d,2n)$.

Proof. Extend $\underline{j'}$ to some $\underline{j}=\{q,r\}\cup\underline{j'}\in I_{d,2n}$, i.e., add two proper indices $q<r$ into $\underline{j'}$. Now, consider $V\in A_{\underline{i}}\cap A_{\underline{j}}$. Let $\boldsymbol{M}=(x_{ij})$ be a $\underline{j}$-standard matrix presentation of V and $\boldsymbol{M}_{\underline{i}}\neq 0$. Denote by $\boldsymbol{M}_{\underline{i}}$ the submatrix formed by the $\underline{i}$-th row of $\boldsymbol{M}$ and similarly for any index in $I_{d,2n}$. Then, $\boldsymbol{N}=\boldsymbol{M}\boldsymbol{M}_{\underline{i}}^{-1}$ is $\underline{i}$-standard. Let the columns of $\boldsymbol{M}$ be $\boldsymbol{c}_1,\boldsymbol{c}_2,\cdots,\boldsymbol{c}_d$ and the columns of $\boldsymbol{N}$ be $\tilde{\boldsymbol{c}}_1,\tilde{\boldsymbol{c}}_2,\cdots,\tilde{\boldsymbol{c}}_d$

$$\boldsymbol{M}=[\boldsymbol{c}_1,\boldsymbol{c}_2,\cdots,\boldsymbol{c}_d]=[\tilde{\boldsymbol{c}}_1,\tilde{\boldsymbol{c}}_2,\cdots,\tilde{\boldsymbol{c}}_d]\boldsymbol{M}_{\underline{i}}.$$

Let $s=\#\{a\in\underline{j}, a\leq q\}$ and $t=\#\{a\in\underline{j}, a\leq r\}$,

$$C(\boldsymbol{M},s,t)=\boldsymbol{c}_s^T\boldsymbol{J}\boldsymbol{c}_t=\left(\sum_{k=1}^{d}x_{i_k,s}\tilde{\boldsymbol{c}}_k{}^T\right)\boldsymbol{J}\left(\sum_{l=1}^{d}x_{i_l,t}\tilde{\boldsymbol{c}}_l\right)=\sum_{l=1}^{d}\sum_{k=1}^{d}x_{i_k,s}x_{i_l,t}C(\boldsymbol{N},k,l).$$

Note that $C(\boldsymbol{N},k,l)=0$ if $k=l$ and $C(\boldsymbol{N},k,l)=-C(\boldsymbol{N},l,k)$ for any k,l. Then,

$$\begin{aligned}C(\boldsymbol{M},s,t)&=\sum_{l=1}^{d}\sum_{k=1,k<l}^{d}(x_{i_k,s}x_{i_l,t}-x_{i_l,s}x_{i_k,t})C(\boldsymbol{N},k,l)\\&=(-1)^{\tau(\underline{j'},j_s,j_t)+\tau(\underline{j'},i_k,i_l)}\frac{p_{\underline{j'}}\cup\{i_k,i_l\}}{p_{\underline{j}}}(V)\cdot C(\boldsymbol{N},k,l).\end{aligned}$$

By Proposition 2.2, since $\boldsymbol{M}$ is $\underline{j}$-standard,

$$C(\boldsymbol{M},s,t)=(-1)^{\tau(\underline{j'},j_s,j_t)}\frac{E_{\underline{j'}}}{p_{\underline{j}}}(V).$$

Additionally, for $\underline{i}$-standard $\boldsymbol{N}$,

$$C(\boldsymbol{N},k,l) = (-1)^{\tau(\underline{i}',i_k,i_l)} \frac{E_{\underline{i}'}}{p_{\underline{i}}}(V).$$

Combining these results, we obtain

$$E_{\underline{j}'} = \sum_{l=1}^{d} \sum_{k=1,k<l}^{d} \frac{E_{\underline{i}\setminus\{i_k,i_l\}}}{p_{\underline{i}}} \cdot (-1)^{k+l+\tau(\underline{j}',i_k,i_l)} p_{\underline{j}'\cup\{i_k,i_l\}}.$$

The proof is complete. □

Corollary 2.1. *On affine open $A_{\underline{i}}$, $\underline{i} \in I_{d,2n}$, symplectic Grassmannian $\mathrm{Gr}^C(d,2n)$ is the set-theoretical intersection of $\mathrm{Gr}(d,2n)$ with hyperplanes $E_{\underline{i}'}/p_{\underline{i}}$, where $\underline{i}'$ takes all $\underline{i}' \in I_{d-2,2n}$ and $\underline{i}' \subset \underline{i}$. These hyperplanes form a minimal generating set of the ideal generated by them in the coordinate ring $k[A_{\underline{i}}]$.*

Proof. It is clear from Proposition 2.3 and that

$$\#\{\underline{i}' \in I_{d-2,2n}, \underline{i}' \subset \underline{i}\} = \binom{d}{2} = codim_{\mathrm{Gr}(d,2n)} \mathrm{Gr}^C(d,2n).$$

The proof is complete. □

2.3 Symplectic flag varieties

Generally, a flag is some nesting sequence of subspaces of linear space k^n for given $s,n \in \mathbb{N}$

$$0 \subset V_1 \subset \cdots \subset V_s \subset k^n,$$

where V_t is a proper subspace of V_{t+1} for any $1 \le t \le s-1$. For any $(i_1,i_2,\cdots,i_s) \in I_{s,n}$, the flag variety $\mathrm{Fl}_n(i_1,i_2,\cdots,i_s)$ is defined as the collection of flags

$$\{\underline{V} = (V_1,V_2,\cdots,V_s) \mid 0 \subset V_1 \subset \cdots \subset V_s \subset k^n, \dim V_t = i_t, \forall 1 \le t \le s\}.$$

If $s=n$, then $\dim V_t = t$ for all $1 \le t \le n$, and such $\mathrm{Fl}_n(1,2,\ldots,n)$ is called a complete flag variety. Otherwise, $\mathrm{Fl}_n(i_1,\cdots,i_s)$ is a partial flag variety.

As with Grassmannian varieties, we can identify SL_{2n}/B^A with the flag variety $\mathrm{Fl}_{2n}(1,2,\ldots,2n)$ and $\mathrm{SL}_{2n}/P^A_{12\cdots n}$ with the partial flag variety $\mathrm{Fl}_{2n}(1,2,\ldots,n)$, where $P^A_{12\cdots n} = P^A_1 \cap P^A_2 \cap \cdots \cap P^A_n$. Since for the Borel subgroup B^C of Sp_{2n}, we have $B^C = B^A \cap \mathrm{Sp}_{2n} = P^A_{12\cdots n} \cap \mathrm{Sp}_{2n}$, we can embed the quotient Sp_{2n}/B^C into

$\mathrm{SL}_{2n}/P^A_{12\cdots n}$, factoring through the canonical surjection $\mathrm{SL}_{2n}/B^A\to\mathrm{SL}_{2n}/P^A_{12\cdots n}$. This correspondence identifies Sp_{2n}/B^C with

$$\{(V_1,V_2,\cdots,V_{2n})\in\mathrm{Fl}_{2n}(1,2,\dots,2n)\,|\,V_t\perp JV_{2n-t},\forall 1\le t\le n\}\subset\mathrm{SL}_{2n}/B^A$$

and also with

$$\{(V_1,V_2,\cdots,V_n)\in\mathrm{Fl}_{2n}(1,2,\dots,n)\,|\,V_n\perp JV_n\}\subset\mathrm{SL}_{2n}/P^A_{12\cdots n}.$$

In this section, we are concerned mainly with the second identification. Denote

$$\mathrm{Sp}_{2n}/B^C=\mathrm{Fl}^C_{2n}(1,2,\dots,n)=\{(V_1,V_2,\cdots,V_n)\in\mathrm{Fl}_{2n}(1,2,\dots,n)\,|\,V_n\perp JV_n\}.$$

Next, let us introduce the homogeneous coordinates and the affine open cover on $\mathrm{Fl}_{2n}(1,2,\dots,n)$. For a projective variety X, let $\Gamma_h(X)$ be its homogeneous coordinate ring and $\Gamma^t_h(X)$ be the t-th grading. For $f\in\Gamma_h(X)$, we say f is a homogeneous section on X. Given projective varieties X,Y and surjective homomorphism of graded rings $\Phi:\Gamma_h(Y)\to\Gamma_h(X)$, there must be a unique morphism of projective varieties $\varphi:X\to Y$ such that the rational functions $\Phi(f)/\Phi(g)=f/g\circ\varphi$ for any $f,g\in\Gamma_h(Y)$ with $\deg f=\deg g$. At this time, we denote $\Phi=\varphi^*$ and for every $f\in\Gamma_h(Y),x\in X$, we note that $\varphi^*(f)(x)=0$ if and only if $f(\varphi(x))=0$. We say such $\varphi:X\to Y$ is induced by $\varphi^*:\Gamma_h(Y)\to\Gamma_h(X)$, but to be intuitive, usually we declare the morphism of varieties φ first. It is easy to check that, in the remainder of this paper, every morphism of projective varieties is in this form. Moreover, if φ is just the inclusion induced by $\varphi^*:\Gamma_h(Y)\to\Gamma_h(Y)/I=\Gamma_h(X)$ for some ideal $I\subset\Gamma_h(Y)$, we also denote $\varphi^*(f)$ by $f|_X$ for $f\in\Gamma_h(Y)$. Sometimes we are concerned with only the certain 1-st grading, which determines the information on every grading, and we simply denote $\varphi^*:\Gamma^1_h(Y)\to\Gamma^1_h(X)$ as well.

$\mathrm{Fl}_{2n}(1,2,\dots,n)$ is a closed subvariety of $\mathrm{Gr}(1,2n)\times\cdots\times\mathrm{Gr}(n,2n)$ [12, p.126], which enables us to describe any homogeneous section in $\Gamma_h(\mathrm{Fl}_{2n}(1,2,\dots,n))$ as a polynomial in $\{p_{\underline{i}}\,|\,\underline{i}\in\cup^n_{d=1}I_{d,2n}\}$, which should be homogeneous in $\{p_{\underline{i}}\,|\,\underline{i}\in I_{d,2n}\}$ for every fixed d. On every Grassmannian $\mathrm{Gr}(d,2n)$, we have defined a series of $E_{\underline{i}'},\underline{i}'\in I_{d-2,2n}$; now, they are also lying in $\Gamma_h(\mathrm{Fl}_{2n}(1,2,\dots,n))$. To avoid confusion about the notation, one should note the cardinal number d in the subscript $\underline{i}$.

For $\underline{V}=(V_1,V_2,\cdots,V_n)\in\mathrm{Fl}_{2n}(1,2,\dots,n)$, we can find a sequence of vectors $\boldsymbol{v}_1,\boldsymbol{v}_2,\cdots,\boldsymbol{v}_n$ such that $V_t=\mathrm{Span}\{\boldsymbol{v}_1,\cdots,\boldsymbol{v}_t\}$ for every $1\le t\le n$. Then $\boldsymbol{M}_d=[\boldsymbol{v}_1,\cdots,\boldsymbol{v}_d]$ for $1\le d\le n$ is a matrix presentation of V_d for $1\le d\le n$, and we say $\boldsymbol{M}_n$ is a matrix presentation of $\underline{V}$. There must be a sequence $(w_1,w_2,\cdots,w_n)$, where w_t are distinct in $\{1,2,\dots,2n\}$, such that $p_{\{w_1,\cdots,w_t\}}(\underline{V})=p_{\{w_1,\cdots,w_t\}}(V_t)\neq 0$. Consider

$$\Delta_{n,2n}=\{(w_1,w_2,\cdots,w_n)\,|\,w_1,w_2,\cdots,w_n\ \text{ are distinct in }\ \{1,2,\dots,2n\}\}$$

and for $\forall \underline{\mathbf{w}} = (w_1, w_2, \cdots, w_n) \in \Delta_{n,2n}$ (note that $\underline{\mathbf{w}}$ is a sequence, can not be rearranged like $\underline{i} \in I_{d,2n}$), let

$$\begin{aligned}&\underline{\mathbf{w}}^{(d)} = \{w_1, w_2, \cdots, w_d\} \in I_{d,2n}, \quad 1 \le d \le n,\\&O_{\underline{\mathbf{w}}} = \{\underline{V} \in \mathrm{Fl}_{2n}(1,2,\ldots,n) | p_{\underline{\mathbf{w}}^{(d)}}(\underline{V}) \neq 0, \ \forall 1 \le d \le n\}.\end{aligned}$$

Then, all the $O_{\underline{\mathbf{w}}}, \underline{\mathbf{w}} \in \Delta_{n,2n}$ form an affine (isomorphic to $\mathbb{A}^{(3n^2-n)/2}$) open cover of $\mathrm{Fl}_{2n}(1,2,\ldots,n)$.

Proposition 2.4. *Let $\underline{\mathbf{w}} \in \Delta_{n,2n}$. If $\underline{i} = \underline{\mathbf{w}}^{(d_1)}$ and $\underline{j} = \underline{\mathbf{w}}^{(d_2)}$ with $d_1 \le d_2$, then $\forall s,t \le d_1$*

$$\frac{E_{\underline{i} \backslash \{i_s, i_t\}}}{p_{\underline{i}}} = \pm \frac{E_{\underline{j} \backslash \{j_{s'}, j_{t'}\}}}{p_{\underline{j}}}$$

on $O_{\underline{\mathbf{w}}}$ for some $s', t' \le d_2$.

Proof. We only need to prove for $d_2 = d_1 + 1$. Consider $\forall \underline{V} \in O_{\underline{\mathbf{w}}}$. There must be an $\underline{i}$-standard matrix presentation of V_{d_1}. Then, we can extend the columns of this matrix to a basis of V_{d_2} by means of a vector v with the t-th component equal to zero for all $t \in \underline{i}$. Additionally, the $\underline{j} \backslash \underline{i}$-th component of v must be nonzero because $p_{\underline{j}}(V_{d_2}) \neq 0$. Up to a column permutation, we obtain a $\underline{j}$-standard matrix presentation of V_{d_2}. By Proposition 2.2, we can complete the proof. □

3 Symplectic Schubert varieties in Grassmannian and flag variety

Let $G = \mathrm{SL}_{2n}$ (actually the following discussions are available for any connected semisimple algebraic group G [18, Chapters 6-8]). We fix a maximal torus T, the diagonal maximal torus, and a standard Borel $B = B^A$ in G. Let $W(G) = N_G(T)/T$, the Weyl group of G with respect to T, we have $W(G) \cong S_{2n}$. We can define the length function $l(\cdot)$ on $W(G)$. We write every permutation in S_{2n} in a one-line expression: for $\sigma \in S_{2n}$, we denote it by $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_{2n})$, where $\sigma_t = \sigma(t)$. For $\theta \in S_{2n}$, the length is $l^A(\theta) = \tau(\theta)$, the inversion number of one-line expression of θ.

For any parabolic subgroup P of G containing B and a given $\theta \in W(G)$ with representative $\dot{\theta} \in N_G(T)$, we denote the well-defined coset $\dot{\theta}P$ in G/P by e_θ. Furthermore, in every coset $\theta W(P) \in W(G)/W(P)$, there exists a unique element in minimal length (in $W(G)$). Collecting all such shortest representatives of cosets, we define

$$W(G)_P^{\min}=\{\theta\in W(G)\,|\,l(\theta)\le l(\theta'),\forall\theta'\in\theta W(P)\}.$$

Take $P=P_d^A$, then the Weyl group

$$W(P_d^A)\cong\{(\sigma(1),\sigma(2),\cdots,\sigma(2n))\in S_{2n}\,|\,\{\sigma(1),\cdots,\sigma(d)\}=\{1,\ldots,d\}\}\cong S_d\times S_{2n-d}.$$

That implies that we can identify $W(\mathrm{SL}_{2n})_{P_d^A}^{\min}$ with $I_{d,2n}$

$$(\sigma(1),\sigma(2),\cdots,\sigma(d),\cdots)\in W(\mathrm{SL}_{2n})_{P_d^A}^{\min}\;\mapsto\;\underline{i}=(\sigma(1),\sigma(2),\cdots,\sigma(d))\in I_{d,2n}.$$

Under this identification, the notation $e_{\underline{i}}:=\underline{i}P_d^A$ is consistent with our definition in Section 1.

B naturally acts on G/P as a subgroup of G, and the B-orbits under this action are determined by representatives $e_\theta:=\theta P$ for all $\theta\in W(G)_P^{\min}$, i.e. the orbits are $Be_\theta,\theta\in W(G)_P^{\min}$.

A Schubert variety $X(\theta)=\overline{Be_\theta},\theta\in W(G)_P^{\min}$ is defined to be the closure of Be_θ in G/P. This closure is also a union of B-orbits. We have the Bruhat order on $W(G)_P^{\min}$ (also $W(G)/W(P)$): for $\theta',\theta\in W(G)_P^{\min}$, if $Be_{\theta'}\subset X(\theta)=\overline{Be_\theta}$ or equivalently $e_{\theta'}\in X(\theta)$, we say $\theta'\le\theta$. Note that G/P itself is also a special Schubert variety.

Similarly, we have the following settings for $G=\mathrm{Sp}_{2n}$ (the maximal torus is always the diagonal torus) (see also [2, Chapter 5]), part of the notations are interpreted after the Table 1.

Table 1: The shortest representatives for different G,B and P.

G	B	$W(G)$	P	$W(G)_P^{\min}$	(general notation for) $\theta\in W(G)_P^{\min}$
SL_{2n}	B^A	S_{2n}	P_d^A	$I_{d,2n}$	$\underline{i}$
			$P_{12\cdots n}^A$	$\Delta_{n,2n}$	$\underline{\mathbf{w}}$
Sp_{2n}	B^C	$\Delta_{n,2n}^{Sp}$	P_d^C	$I_{d,2n}^{Sp}$	$\underline{i}$
			B^C	$\Delta_{n,2n}^{Sp}$	$\underline{\mathbf{w}}$

Here,

$$W(\mathrm{Sp}_{2n})=\{(\sigma_1,\sigma_2,\cdots,\sigma_{2n})\in S_{2n}\,|\,\sigma_t+\sigma_{2n+1-t}=2n+1,\forall 1\le t\le n\},$$

it can be identified with

$$\Delta_{n,2n}^{Sp}=\{(w_1,w_2,\cdots,w_n)\in\Delta_{n,2n}\,|\,w_s+w_t\ne 2n+1,\forall s\ne t\}\subset\Delta_{n,2n}.$$

Moreover, the length function for Sp_{2n} is given by

$$l^C(\theta)=\frac{\tau(\theta)+m}{2},\quad m=\#\{1\leq t\leq n\,|\,\theta_t>n\},\quad \forall\theta\in W(\mathrm{Sp}_{2n}).$$

The Bruhat order on $I_{d,2n}$ (resp. $I^{Sp}_{d,2n}$) is given by (cf. [2, Chapter 5]): for

$$\underline{i}=(i_1,i_2,\cdots,i_d),\quad \underline{j}=(j_1,j_2,\cdots,j_d)\in I_{d,2n}\quad (\text{resp. } I^{Sp}_{d,2n})$$

with $i_1<i_2<\cdots<i_d,j_1<j_2<\cdots<j_d$, we have $\underline{i}\leq\underline{j}$ under the Bruhat order if and only if $i_t\leq j_t$ for every $1\leq t\leq d$. Additionally, the Bruhat order on $\Delta_{n,2n}$ (resp. $\Delta^{Sp}_{n,2n}$) is: for $\underline{\mathbf{w}},\underline{\mathbf{w}'}\in\Delta_{n,2n}$ (resp. $\Delta^{Sp}_{n,2n}$), $\underline{\mathbf{w}}\leq\underline{\mathbf{w}'}$ if and only if $\underline{\mathbf{w}}^{(d)}\leq\underline{\mathbf{w}'}^{(d)}$ in every $I_{d,2n},1\leq d\leq n$.

Moreover, in the language of subspaces, we can characterize the Schubert varieties in $\mathrm{SL}_{2n}/P^A_d=\mathrm{Gr}(d,2n)$ as for any $\underline{i}\in I_{d,2n}$,

$$X^A(\underline{i})=\big\{V\in\mathrm{Gr}(d,2n)\,|\,\dim(V\cap\mathrm{Span}\{e_1,e_2,\cdots,e_{i_t}\})\geq t,\forall 1\leq t\leq d\big\}.$$

The Schubert varieties in flag variety $\mathrm{SL}_{2n}/P^A_{12\cdots n}=\mathrm{Fl}_{2n}(1,2,\dots,n)$ are such that for any $\underline{\mathbf{w}}\in\Delta_{n,2n}$,

$$X^A(\underline{\mathbf{w}})=\big\{\underline{V}\in\mathrm{Fl}_{2n}(1,2,\dots,n)\,|\,V_d\in X^A(\underline{\mathbf{w}}^{(d)}),\forall 1\leq d\leq n\big\}.$$

Lemma 3.1 ([2, p.16], [12, p.173]). *Let $\underline{i}\in I_{d,2n}$. The defining ideal of $X^A(\underline{i})$ in the homogeneous coordinate ring of $\mathrm{Gr}(d,2n)$ is generated by*

$$\big\{p_{\underline{j}}\,|\,\underline{j}\in I_{d,2n},\underline{j}\not\leq\underline{i}\big\}.$$

Let $\underline{\mathbf{w}}\in\Delta_{n,2n}$. The defining ideal of $X^A(\underline{\mathbf{w}})$ in the homogeneous coordinate ring of $\mathrm{Fl}_{2n}(1,2,\dots,n)$ is generated by

$$\bigcup_{d=1}^{n}\big\{p_{\underline{j}}\,|\,\underline{j}\in I_{d,2n},\underline{j}\not\leq\underline{\mathbf{w}}^{(d)}\big\}.$$

The symplectic Schubert varieties in the Grassmannian and flag variety, respectively, are for any $\underline{i}\in I^{Sp}_{d,2n}\subset I_{d,2n}$

$$X^C(\underline{i})=X^A(\underline{i})\cap\mathrm{Gr}^C(d,2n),$$

and for any $\underline{\mathbf{w}}\in\Delta^{Sp}_{n,2n}\subset\Delta_{n,2n}$

$$X^C(\underline{\mathbf{w}})=X^A(\underline{\mathbf{w}})\cap\mathrm{Sp}_{2n}/B^C.$$

We should be careful that even though $I_{d,2n}^{Sp} \subset I_{d,2n}$ and there are identifications $W(\mathrm{Sp}_{2n})_{P_d^C}^{\min} = I_{d,2n}^{Sp}$, $W(\mathrm{SL}_{2n})_{P_d^A}^{\min} = I_{d,2n}$, in general, an index $\underline{i} \in I_{d,2n}^{Sp}$ corresponds to two different representatives in $W(\mathrm{Sp}_{2n})_{P_d^C}^{\min}$ and $W(\mathrm{SL}_{2n})_{P_d^A}^{\min}$, as we will discuss in Section 4. However, viewing $I_{d,2n}^{Sp}$ the index set as a subset of $I_{d,2n}$ provides many conveniences, it only means that for any $\underline{i} \in I_{d,2n}^{Sp}$, it can be used as an index for both Schubert varieties of types A and C in the Grassmannian (similarly in flag varieties).

Naturally, for every $d \leq n$, there is a projection

$$pr_d : \mathrm{Fl}_{2n}(1,2,\ldots,n) \;\to\; \mathrm{Gr}(d,2n), \quad \underline{V} = (V_1, V_2, \cdots, V_n) \;\mapsto\; V_d.$$

In a flag variety, a flag $\underline{V} \in \mathrm{Fl}_{2n}(1,2,\ldots,n)$ belongs to Sp_{2n}/B^C if and only if $pr_n(\underline{V}) \in \mathrm{Gr}^C(n,2n)$. By Corollary 2.1, we immediately obtain the following.

Theorem 3.1. *Let $\underline{i} \in I_{d,2n}^{Sp}$ (resp. $\underline{\mathbf{w}} \in \Delta_{n,2n}^{Sp}$). The symplectic Schubert variety $X^C(\underline{i})$ (resp. $X^C(\underline{\mathbf{w}})$) is the intersection of $X^A(\underline{i})$ (resp. $X^A(\underline{\mathbf{w}})$) with hyperplanes $E_{\underline{i'}}$, $\underline{i'} \in I_{d-2,2n}$ (resp. $\underline{i'} \in \cup_{d'=2}^n I_{d'-2,2n}$).*

However, it is not sufficient to say that $\{E_{\underline{i'}} \mid \underline{i'} \in I_{d-2,2n}\}$ generates the defining ideal $I_{X^A(\underline{i})}(X^C(\underline{i}))$ of $X^C(\underline{i})$ in $X^A(\underline{i})$, $\underline{i} \in I_{d,2n}^{Sp}$

$$I_{X^A(\underline{i})}\left(X^C(\underline{i})\right) = \left\{ f \in \Gamma_h\left(X^A(\underline{i})\right) \mid f(V) = 0, \forall V \in X^C(\underline{i}) \right\}.$$

Even though $X^C(\underline{i})$ is irreducible, we can say only that $I_{X^A(\underline{i})}(X^C(\underline{i}))$ is the radical of the ideal generated by $\{E_{\underline{i'}} \mid \underline{i'} \in I_{d-2,2n}\}$. To solve this problem, we need more preparation.

Our goal here is the following theorem.

Theorem 3.2. *On a Grassmannian variety, for $\underline{i} \in I_{d,2n}^{Sp}$, the defining ideal of $X^C(\underline{i})$ in $X^A(\underline{i})$ is generated by $\{E_{\underline{i'}} \mid \underline{i'} \in I_{d-2,2n}\}$. On a flag variety, for $\underline{\mathbf{w}} \in \Delta_{d,2n}^{Sp}$, the defining ideal of $X^C(\underline{\mathbf{w}})$ in $X^A(\underline{\mathbf{w}})$ is generated by $\cup_{d=2}^n \{E_{\underline{i'}} \mid \underline{i'} \in I_{d-2,2n}\}$.*

Firstly, according to De Concini's work [4], we need to study only the degree-1 graded $\Gamma_h^1(X^A(\underline{i}))$, $\underline{i} \in I_{d,2n}$. Specifically, De Concini provided an algorithm [4, (2.2),(2.4)] (see also [17, p.10]) that modulo some linear relations [4, (1.8)], which are identically zero on $\mathrm{Gr}^C(d,2n)$ (resp. on Sp_{2n}/P_d^C), every homogeneous section $f \in \Gamma_h(\mathrm{Gr}(d,2n))$ (resp. $\Gamma_h(\mathrm{Fl}_{2n}(1,2,\ldots,n))$) is equivalent to a linear combination of "opposite symplectic standard tableaux". The symplectic standard tableaux are homogeneous sections constructed by De Concini [4, (2.3)]. Here, we use the word "opposite" because in his definition, De Concini used the lower triangular Borel, not the upper triangular one. By a conjugate action of an anti-diagonal matrix, we can connect the lower triangular and upper triangular cases, so through an automorphism of homogeneous coordinate rings that induces such action of the anti-diagonal matrix, we can obtain the isomorphic images of De Concini's symplectic standard tableaux: we call them the opposite symplectic standard tableaux. It is not necessary to further explore this definition. We only need the following property of such homogeneous sections here.

Proposition 3.1 ([4, (3.5)]). *A linear combination of opposite symplectic standard tableaux is zero on some $X^C(\underline{i})$, $\underline{i}\in I^{Sp}_{d,2n}$ (resp. $X^C(\underline{\mathbf{w}})$, $\underline{\mathbf{w}}\in\Delta^{Sp}_{n,2n}$) implies that this linear combination is zero on the whole $X^A(\underline{i})$ (resp. $X^A(\underline{\mathbf{w}})$).*

By the algorithm [4, (2.2),(2.4)], for a homogeneous $f\in\Gamma_h(X^A(\underline{i}))$, it can be written as two parts

$$\begin{aligned} f=&\big(f_I=\text{generated by linear sections that are identically zero on }\mathrm{Gr}^C(d,2n)\big)|_{X^A(\underline{i})}\\ &+\big(f_{II}=\text{linear combination of opposite symplectic standard tableaux}\big)|_{X^A(\underline{i})}. \end{aligned}$$

If moreover $f\in I_{X^A(\underline{i})}(X^C(\underline{i}))$ or equivalently $f(V)=0$ for any $V\in X^C(\underline{i})$, then $f_{II}|_{X^C(\underline{i})}=0$. By the above proposition, we immediately obtain $f_{II}|_{X^A(\underline{i})}=0$. Thus, $f=f_I|_{X^A(\underline{i})}$, which implies $I_{X^A(\underline{i})}(X^C(\underline{i}))$ is generated by some restrictions of linear sections in $\Gamma^1_h(\mathrm{Gr}(d,2n))$ on $X^A(\underline{i})$. A similar argument is valid for the Schubert varieties on the flag variety: $I_{X^A(\underline{\mathbf{w}})}(X^C(\underline{\mathbf{w}}))$ is generated by some restrictions of linear sections in $\Gamma^1_h(\mathrm{Fl}_{2n}(1,2,\ldots,n))$ on $X^A(\underline{\mathbf{w}})$, $\forall\underline{\mathbf{w}}\in\Delta^{Sp}_{n,2n}$. Therefore we have the following result.

Proposition 3.2 ([4]). *The defining ideal*

$$I_{X^A(\underline{i})}\big(X^C(\underline{i})\big),\quad \underline{i}\in I^{Sp}_{d,2n}$$

(resp. $I_{X^A(\underline{\mathbf{w}})}(X^C(\underline{\mathbf{w}}))$, $\underline{\mathbf{w}}\in\Delta^{Sp}_{d,2n}$) is generated by the restrictions of some linear (i.e., degree-1) homogeneous sections in $\Gamma^1_h(X^A(\underline{i}))$ on $X^C(\underline{i})$ (resp. by the restrictions of some linear homogeneous sections in $\Gamma^1_h(X^A(\underline{\mathbf{w}}))$ on $X^C(\underline{\mathbf{w}})$).

So for the Schubert varieties in Grassmannian, in order to obtain the goal theorem mentioned above, it is sufficient to prove that every degree-1 homogeneous section in $I_{\mathrm{Gr}(d,2n)}(\mathrm{Gr}^C(d,2n))$ can be spanned by $\{E_{\underline{i'}}\,|\,\underline{i'}\in I_{d-2,2n}\}$, that is

$$\Gamma^1_h\big(\mathrm{Gr}(d,2n)\big)\cap I_{\mathrm{Gr}(d,2n)}\big(\mathrm{Gr}^C(d,2n)\big)=\mathrm{Span}\big\{E_{\underline{i'}}\,|\,\underline{i'}\in I_{d-2,2n}\big\}.$$

In addition, since the projection pr_d is surjective for every $1 \le d \le n$ and the degree-1 sections on flag varieties are exactly from the degree-1 sections on different Grassmannians, to prove the theorem above, we only need to work on Grassmannians.

Recall that we work on *char* $k=0$.

Lemma 3.2. *For $n=2m$, an even integer, let $\underline{i},\underline{j} \in I^{Sp}_{n,2n}$,*

$$\begin{aligned}\underline{i}&=(i_1,i_2,\cdots,i_m,2n+1-i_m,\cdots,2n+1-i_2,2n+1-i_1),\\ \underline{j}&=(j_1,j_2,\cdots,j_m,2n+1-j_m,\cdots,2n+1-j_2,2n+1-j_1)\end{aligned}$$

with $\{i_1,i_2,\cdots,i_m\}\cup\{j_1,j_2,\cdots,j_m\}=\{1,2,\ldots,n\}$. Then

$$(-1)^{m-1}p_{\underline{i}}+p_{\underline{j}}=\sum_{a=0}^{m-1}\sum_{\substack{\underline{l}\\ \#(\underline{l}\cap\underline{j})=2a.}}\frac{(-1)^a}{m\binom{m-1}{a}}E_{\underline{l}},$$

where

$$\underline{l}=(l_1,\cdots,l_{m-1},2n+1-l_{m-1},\cdots,2n+1-l_1)\in I^{Sp}_{n-2,2n}.$$

Proof. Note that in such $E_{\underline{l}}$, the inversion numbers in front of the Plücker coordinates are all $(-1)^{m-1}$. Thus, this lemma can be deduced from a direct computation on the coefficient of every term $p_{(k_1,\cdots,k_m,2n+1-k_m,\cdots,2n+1-k_1)}$ on the right side with $\#(\{k_1,\cdots,k_m,2n+1-k_m,\cdots,2n+1-k_1\}\cap\underline{j})=b$: it is for $b=0$

$$m\cdot\frac{(-1)^0}{m\binom{m-1}{0}}(-1)^{m-1}=(-1)^{m-1},$$

for $b=2m$

$$m\cdot\frac{(-1)^{m-1}}{m\binom{m-1}{m-1}}(-1)^{m-1}=(-1)^{2(m-1)}=1,$$

and for any other b

$$(-1)^{m-1+b}\left(\frac{m-b}{m\binom{m-1}{b}}-\frac{b}{m\binom{m-1}{b-1}}\right)=0.$$

The proof is complete. $\square$

The lemma above shows that for even integer n, if we partition $\{1,2,\ldots,n\}$ into two half parts $\{i_1,i_2,\cdots,i_m\}\cup\{j_1,j_2,\cdots,j_m\}$ and put

$$\underline{i}=(i_1,i_2,\cdots,i_m,2n+1-i_m,\cdots,2n+1-i_2,2n+1-i_1),$$
$$\underline{j}=(j_1,j_2,\cdots,j_m,2n+1-j_m,\cdots,2n+1-j_2,2n+1-j_1),$$

then $p_{\underline{i}}\pm p_{\underline{j}}$ is a linear combination of $\{E_{\underline{i'}}\,|\,\underline{i'}\in I_{d-2,2n}\}$.

We require a simple result of the standard monomial theory on Grassmannians here.

Theorem 3.3 ([12, p.171], [17, p.12]). *$\{p_{\underline{j}}\,|\,\underline{j}\in I_{d,2n}\}$ forms a basis for $\Gamma_h^1(\mathrm{Gr}(d,2n))$. Additionally, $\{p_{\underline{j}}\,|\,\underline{i}\in I_{d,2n},\underline{j}\le\underline{i}\}$ forms a basis for $\Gamma_h^1(X^A(\underline{i})),\forall\underline{i}\in I_{d,2n}$.*

If $d=0$ or 1, we say $E_{\underline{i'}}=0,\forall\underline{i'}\in' I'_{d-2,2n}$ for convenience. For any $1\le d\le n,\underline{i}\in I_{d,2(n-1)}$, we naturally denote $\underline{i+1}=(i_1+1,i_2+1,\cdots,i_d+1)\in I_{d,2n}$.

Theorem 3.4. $\Gamma_h^1(\mathrm{Gr}(d,2n))\cap I_{\mathrm{Gr}(d,2n)}(\mathrm{Gr}^C(d,2n))=\mathrm{Span}\{E_{\underline{i'}}\,|\,\underline{i'}\in I_{d-2,2n}\}$.

Proof. Clearly, for $d=0,1,\forall n$. Let us use the induction on d and n. For $d>1,n\ge d$, assume that

$$\Gamma_h^1\big(\mathrm{Gr}(d',2n)\big)\cap I_{\mathrm{Gr}(d',2n')}\big(\mathrm{Gr}^C(d',2n')\big)=\mathrm{Span}\{E_{\underline{i''}}\,|\,\underline{i''}\in I_{d'-2,2n'}\}$$

for any $d'\le d,d'\le n'\le n,(d',n')\ne(d,n)$. Consider

$$\sum_{\underline{i}\in\Lambda}c_{\underline{i}}p_{\underline{i}}\in\Gamma_h^1\big(\mathrm{Gr}(d,2n)\big)\cap I_{\mathrm{Gr}(d,2n)}\big(\mathrm{Gr}^C(d,2n)\big),$$

where $c_{\underline{i}}\in k$ are coefficients and Λ is an index set. Equivalently,

$$\sum_{\underline{i}\in\Lambda}c_{\underline{i}}p_{\underline{i}}|_{\mathrm{Gr}^C(d,2n)}=0.$$

We want to show that such $\sum c_{\underline{i}}p_{\underline{i}}$ can be written as a linear combination of $\{E_{\underline{i'}}|\underline{i'}\in I_{d-2,2n}\}$.

Put $2n\times 2n$ matrix $\boldsymbol{D}_\lambda=\mathbf{diag}(\lambda,1,1,\cdots,1,\lambda^{-1})$ for $\lambda\in k$. $\boldsymbol{D}_\lambda$ naturally acts on $\mathrm{Gr}(d,2n)$ and sends $\mathrm{Gr}^C(d,2n)$ to $\mathrm{Gr}^C(d,2n)$. It is induced by automorphism on $\Gamma_h(\mathrm{Gr}(d,2n))$ sending

$$\begin{array}{lll} p_{\underline{i}} & \mapsto & \lambda p_{\underline{i}}, \quad \text{if} \quad 1\in\underline{i}, \quad 2n\notin\underline{i}, \\ p_{\underline{i}} & \mapsto & \lambda^{-1}p_{\underline{i}}, \quad \text{if} \quad 1\notin\underline{i}, \quad 2n\in\underline{i}, \\ p_{\underline{i}} & \mapsto & p_{\underline{i}}, \quad \text{if} \quad (1,2n\in\underline{i}) \quad \text{or} \quad (1,2n\notin\underline{i}). \end{array}$$

Additionally, this automorphism of ring maps $I_{\mathrm{Gr}(d,2n)}(\mathrm{Gr}^C(d,2n))$ to itself. Thus, the image

$$\left(\lambda\sum_{\underline{i}\in\Lambda_1}c_{\underline{i}}p_{\underline{i}}+\lambda^{-1}\sum_{\underline{i}\in\Lambda_2}c_{\underline{i}}p_{\underline{i}}+\sum_{\underline{i}\in\Lambda_3}c_{\underline{i}}p_{\underline{i}}\right)\Big|_{\mathrm{Gr}^C(d,2n)}=0,$$

where

$$\begin{aligned} &\Lambda_1=\{\underline{i}\in\Lambda\,|\,1\in\underline{i},2n\notin\underline{i}\},\\ &\Lambda_2=\{\underline{i}\in\Lambda\,|\,1\notin\underline{i},2n\in\underline{i}\},\\ &\Lambda_3=\{\underline{i}\in\Lambda\,|\,(1,2n\in\underline{i}) \;\text{ or }\; (1,2n\notin\underline{i})\}. \end{aligned}$$

By the arbitrariness of λ, we have the following three parts:

$$\sum_{\underline{i}\in\Lambda_1}c_{\underline{i}}p_{\underline{i}}|_{\mathrm{Gr}^C(d,2n)}=\sum_{\underline{i}\in\Lambda_2}c_{\underline{i}}p_{\underline{i}}|_{\mathrm{Gr}^C(d,2n)}=\sum_{\underline{i}\in\Lambda_3}c_{\underline{i}}p_{\underline{i}}|_{\mathrm{Gr}^C(d,2n)}=0.$$

Thus, we only need the proof for the three types $\Lambda=\Lambda_1$, $\Lambda=\Lambda_2$ and $\Lambda=\Lambda_3$.

Type 1. $\Lambda=\Lambda_1.\forall\underline{i}\in\Lambda,1\in\underline{i}$ and $2n\notin\underline{i}$.

Consider the embedding $\varphi_1:\mathrm{Gr}(d-1,2(n-1))\to\mathrm{Gr}(d,2n)$, which maps $V\in\mathrm{Gr}(d-1,2(n-1))$ with a matrix presentation $\boldsymbol{M}$ to $\varphi_1(V)\in\mathrm{Gr}(d,2n)$ with matrix presentation $\boldsymbol{M}_1$ as follows:

$$V=\mathrm{Span}\{\text{columns of }\boldsymbol{M}\}\;\mapsto\;\varphi(V)=\mathrm{Span}\left\{\text{columns of }\boldsymbol{M}_1=\begin{bmatrix}1 & 0\\ \mathbf{0}_{2(n-1)\times 1} & \boldsymbol{M}\\ 0 & 0\end{bmatrix}\right\}.$$

It is induced by $\varphi_1^*:\Gamma_h^1(\mathrm{Gr}(d,2n))\to\Gamma_h^1(\mathrm{Gr}(d-1,2(n-1)))$ sending

$$p_{\underline{j+1}\cup\{1\}}\;\mapsto\;p_{\underline{j}},\quad\forall\underline{j}\in I_{d-1,2(n-1)}$$

and having a kernel spanned by

$$\left\{p_{\underline{j}}\,|\,\underline{j}\in I_{d,2n},2n\in\underline{j}\ \text{ or }\ 1\notin\underline{j}\right\}.$$

For $V\in\mathrm{Gr}^C(d-1,2(n-1))$, $\varphi_1(V)\in\mathrm{Gr}^C(d,2n)$. Then,

$$\begin{aligned}&\varphi_1^*\Big(\sum_{\underline{i}\in\Lambda_1}c_{\underline{i}}p_{\underline{i}}\Big)\Big|_{\mathrm{Gr}^C(d-1,2(n-1))}=0,\\&\varphi_1^*\Big(\sum_{\underline{i}\in\Lambda_1}c_{\underline{i}}p_{\underline{i}}\Big)\in I_{\mathrm{Gr}(d-2,2(n-1))}\big(\mathrm{Gr}^C(d-2,2(n-1))\big).\end{aligned}$$

From our assumption on $\mathrm{Gr}^C(d-1,2(n-1))$, we have

$$\varphi_1^*\Big(\sum_{\underline{i}\in\Lambda_1}c_{\underline{i}}p_{\underline{i}}\Big)=\sum_{\underline{i''}\in I_{d-3,2(n-1)}}c_{\underline{i''}}E_{\underline{i''}}.$$

Note that one preimage of $\sum c_{\underline{i''}}E_{\underline{i''}}$ under φ_1^* is $\sum c_{\underline{i''}}E_{\underline{i''+1}\cup\{1\}}$ (or zero, if $d-1\leq 1$). So

$$\sum_{\underline{i}\in\Lambda_1}c_{\underline{i}}p_{\underline{i}}-\sum_{\underline{i''}\in I_{d-3,2(n-1)}}c_{\underline{i''}}E_{\underline{i''}\cup\{1\}}\in\ker\varphi_1^*.$$

However, for every $\underline{i}$ of $p_{\underline{i}}$ appearing in $\sum_{\underline{i}\in\Lambda_1}c_{\underline{i}}p_{\underline{i}}-\sum_{\underline{i''}\in I_{d-3,2(n-1)}}c_{\underline{i''}}E_{\underline{i''}\cup\{1\}}$, there must be $1\in\underline{i},2n\notin\underline{i}$. $\ker\varphi_1^*$ is spanned by $\{p_{\underline{j}},\underline{j}\in I_{d-1,2(n-1)}\,|\,2n\in\underline{j}\text{ or }1\notin\underline{j}\}$. Because of Theorem 3.3, there must be

$$\sum_{\underline{i}\in\Lambda_1}c_{\underline{i}}p_{\underline{i}}-\sum_{\underline{i''}\in I_{d-3,2(n-1)}}c_{\underline{i''}}E_{\underline{i''}\cup\{1\}}=0$$

in $\Gamma_h^1(\mathrm{Gr}(d,2n))$.

Type 2. $\Lambda=\Lambda_2$. $\forall\underline{i}\in\Lambda,1\notin\underline{i}$ and $2n\in\underline{i}$.

Consider the embedding $\mathrm{Gr}(d-1,2(n-1))\to\mathrm{Gr}(d,2n)$ defined as

$$V=\mathrm{Span}\{\text{columns of }\boldsymbol{M}\}\ \mapsto\ \varphi(V)=\mathrm{Span}\left\{\text{columns of }\boldsymbol{M}_1=\begin{bmatrix}0&0\\\boldsymbol{M}&\mathbf{0}_{2(n-1)\times1}\\0&1\end{bmatrix}\right\}.$$

It is induced by $\Gamma_h^1(\mathrm{Gr}(d,2n))\to\Gamma_h^1(\mathrm{Gr}(d-1,2(n-1)))$ sending

$$p_{\underline{j+1}\cup\{2n\}}\ \mapsto\ p_{\underline{j}},\quad\forall\underline{j}\in I_{d-1,2(n-1)}$$

and having a kernel spanned by

$$\left\{p_{\underline{j}} \,|\, \underline{j}\in I_{d,2n},\, 1\in\underline{j} \text{ or } 2n\notin\underline{j}\right\}.$$

Then, similar to Type 1, we can express such a Type 2 linear section as a linear combination of $\{E_{\underline{i'}} \,|\, \underline{i'}\in I_{d-2,2n}\}$.

Type 3. $\Lambda=\Lambda_3$. $\forall \underline{i}\in\Lambda$, both of $1,2n\in\underline{i}$ or $1,2n\notin\underline{i}$.

We have two cases: $d<n$ or $d=n$. If $d<n$, consider the embedding $\varphi_2:\mathrm{Gr}(d,2(n-1))\to\mathrm{Gr}(d,2n)$ defined as

$$V=\mathrm{Span}\{\text{columns of } \boldsymbol{M}\} \;\mapsto\; \varphi_2(V)=\mathrm{Span}\left\{\text{columns of } \begin{bmatrix}\mathbf{0}_{1\times d}\\ \boldsymbol{M}\\ \mathbf{0}_{1\times d}\end{bmatrix}\right\}.$$

It is induced by $\varphi_2^*:\Gamma_h^1(\mathrm{Gr}(d,2n))\to\Gamma_h^1(\mathrm{Gr}(d,2(n-1)))$ sending

$$p_{\underline{j+1}} \;\mapsto\; p_{\underline{j}}, \quad \forall \underline{j}\in I_{d,2(n-1)}$$

and having a kernel spanned by

$$\left\{p_{\underline{j}} \,|\, \underline{j}\in I_{d,2n},\, 1\in\underline{j} \text{ or } 2n\in\underline{j}\right\}.$$

Now, there must be

$$\varphi_2^*\left(\sum_{\underline{i}\in\Lambda} c_{\underline{i}}p_{\underline{i}}\right)\in\Gamma_h^1\big(\mathrm{Gr}(d,2(n-1))\big)\cap I_{\mathrm{Gr}(d,2(n-1))}\big(\mathrm{Gr}^C(d,2(n-1))\big).$$

By our assumption

$$\varphi_2^*\left(\sum_{\underline{i}\in\Lambda} c_{\underline{i}}p_{\underline{i}}\right)=\sum_{\underline{i''}\in I_{d-2,2(n-1)}} c_{\underline{i''}}E_{\underline{i''}}.$$

Similarly as in Type 1, one preimage of $\sum c_{\underline{i''}}E_{\underline{i''}}$ under φ_2^* is $\sum c_{\underline{i''}}E_{\underline{i''+1}}$. Thus,

$$\sum_{\underline{i}\in\Lambda} c_{\underline{i}}p_{\underline{i}}-\sum_{\underline{i''}\in I_{d-2,2(n-1)}} c_{\underline{i''}}E_{\underline{i''+1}}\in\ker\varphi_2^*.$$

Here, $\ker\varphi_2^*$ is spanned by $\{p_{\underline{j}} \,|\, \underline{j}\in I_{d,2n}, 1\in\underline{j} \text{ or } 2n\in\underline{j}\}$. Comparing the subscripts, we have

$$\sum_{\underline{i}\in\Lambda} c_{\underline{i}}p_{\underline{i}}-\sum_{\underline{i''}\in I_{d-2,2(n-1)}} c_{\underline{i''}}E_{\underline{i''+1}}=\sum_{\underline{j}\in\Lambda_4} c_{\underline{j}}p_{\underline{j}},$$

where $\forall \underline{j}\in\Lambda_4\subset I_{d,2n}$, both of $1,2n\in\underline{j}$. Now, we must prove that $\sum_{\underline{i}\in\Lambda_4} c_{\underline{j}}p_{\underline{j}}$ is a linear combination of $\{E_{\underline{i'}} \,|\, \underline{i}\in I_{d-2,2n}\}$. We leave this as a lemma.

In the case $d=n$ and for Type 3 $\sum_{\underline{i}\in\Lambda} c_{\underline{i}} p_{\underline{i}}$, for any $1\leq t\leq n$, we have also

$$\sum_{\substack{\underline{i}\in\Lambda\\ t\in\underline{i}\\ 2n+1-t\notin\underline{i}}} c_{\underline{i}} p_{\underline{i}}|_{\mathrm{Gr}^C(d,2n)} = \sum_{\substack{\underline{i}\in\Lambda\\ t\notin\underline{i}\\ 2n+1-t\in\underline{i}}} c_{\underline{i}} p_{\underline{i}}|_{\mathrm{Gr}^C(d,2n)} = 0.$$

Therefore, they are linear combinations of $\{E_{\underline{i'}}\,|\,\underline{i'}\in I_{d-2,2n}\}$ through an argument similar to our proof for Types 1 and 2 above. Therefore, after repeatedly excluding such terms, we have

$$\sum_{\underline{i}\in\Lambda} c_{\underline{i}} p_{\underline{i}} - (\text{a linear combination of } \{E_{\underline{i'}}\,|\,\underline{i'}\in I_{d-2,2n}\}) = \sum_{\underline{i}\in\Lambda_4'} c_{\underline{i}} p_{\underline{i}},$$

where for $\forall\,\underline{i}\in\Lambda_4', \forall\,1\leq t\leq n$, $t,2n+1-t\in\Lambda_4'$ or $t,2n+1-t\notin\Lambda_4'$. If $d=n$ is odd, the proof is complete. If $d=n=2m$ is even, we still need to show $\sum_{\underline{i}\in\Lambda_4'} c_{\underline{i}} p_{\underline{i}}$ is a linear combination of $\{E_{\underline{i'}}\,|\,\underline{i'}\in I_{d-2,2n}\}$. Note that by subtracting some relations in Lemma 3.2, which are linear combinations of $\{E_{\underline{i'}}\,|\,\underline{i'}\in I_{d-2,2n}\}$, we have

$$\sum_{\underline{i}\in\Lambda_4'} c_{\underline{i}} p_{\underline{i}} - (\text{a linear combination of } \{E_{\underline{i'}}\,|\,\underline{i'}\in I_{d-2,2n}\}) = \sum_{\underline{i}\in\Lambda_4} c_{\underline{i}} p_{\underline{i}},$$

where $\forall\underline{i}\in\Lambda_4$, both $1,2n\in\underline{i}$. Additionally, we have the condition $\sum_{\underline{i}\in\Lambda_4} c_{\underline{i}} p_{\underline{i}}|_{\mathrm{Gr}^C(d,2n)} = 0$.

Thus, our proof is completed by the following lemma. $\square$

Remark 3.1. For any $\underline{i}\in I_{d,2n}$, the embedding φ_1 gives an isomorphism $X^A(\underline{i})\to X^A(\underline{i+1}\cup\{1\})$, which also sends $X^C(\underline{i})\to X^C(\underline{i+1}\cup\{1\})$ if $\underline{i}\in I_{d,2n}^{Sp}$.

Lemma 3.3. *Let $d>1, n\geq d$. If*

$$\Gamma_h^1(\mathrm{Gr}(d',2n))\cap I_{\mathrm{Gr}(d',2n')}(\mathrm{Gr}^C(d',2n')) = \mathrm{Span}\{E_{\underline{i''}}\,|\,\underline{i''}\in I_{d'-2,2n'}\}$$

for any $d'\leq d, d'\leq n'\leq n, (d',n')\neq(d,n)$ (the induction assumption in above proof), and

$$\sum_{\underline{i}\in\Lambda} c_{\underline{i}} p_{\underline{i}}|_{\mathrm{Gr}^C(d,2n)} = 0,$$

where $\forall\,\underline{i}\in\Lambda$, both of $1,2n\in\underline{i}$, then $\sum_{\underline{i}\in\Lambda} c_{\underline{i}} p_{\underline{i}}$ is a linear combination of $\{E_{\underline{i}}\,|\,\underline{i}\in I_{d-2,2n}\}$.

Proof. Consider the embedding $\varphi_3 : \mathrm{Gr}(d-2, 2(n-1)) \to \mathrm{Gr}(d, 2n)$

$$V = \mathrm{Span}\{\text{columns of } \boldsymbol{M}\} \ \mapsto$$

$$\varphi_3(V) = \mathrm{Span}\left\{ \text{columns of } \begin{bmatrix} 1 & \mathbf{0}_{1\times(d-2)} & 0 \\ \mathbf{0}_{2(n-1)\times 1} & \boldsymbol{M} & \mathbf{0}_{2(n-1)\times 1} \\ 0 & \mathbf{0}_{1\times(d-2)} & 1 \end{bmatrix} \right\}.$$

It is induced by $\varphi_3^* : \Gamma_h^1(\mathrm{Gr}(d,2n)) \to \Gamma_h^1(\mathrm{Gr}(d-2, 2(n-1)))$ sending

$$\begin{aligned} p_{\underline{i'}\cup 1,2n} &\mapsto p_{\underline{i'}}, \quad \forall \underline{i'} \in I_{d-2,2n}, \\ p_{\underline{i}} &\mapsto 0, \quad\ \ \forall \underline{i} \in I_{d,2n} \ \text{ s.t. } 1 \notin \underline{i} \ \text{ or } \ 2n \notin \underline{i}. \end{aligned}$$

We should be careful that $\varphi_3(\mathrm{Gr}^C(d-2, 2(n-1))) \neq \mathrm{Gr}^C(d, 2n)$. However, for any $V \in \mathrm{Gr}^C(d-2, 2(n-1))$ with matrix presentation $M_{2(n-1)\times(d-2)}$, since $d-2 \le n-2$ $= (n-1)-1$, we can extend V to a $(n-2)$-dimensional subspace V' of $k^{2(n-1)}$ with $V' \perp \boldsymbol{J}V'$. The orthogonal complement of $\boldsymbol{J}V'$ in $k^{2(n-1)}$ is of dimension $2(n-1)-(n-2) = n = (n-1)+1$, so we can find two linearly independent vectors $\boldsymbol{m}_1, \boldsymbol{m}_2 \in k^{2(n-1)}$ such that (here $\boldsymbol{J}$ is $(d-2)\times(d-2)$ standard symplectic matrix)

$$\boldsymbol{m}_1^T \boldsymbol{J} \boldsymbol{M} = \boldsymbol{m}_2^T \boldsymbol{J} \boldsymbol{M} = 0, \quad \boldsymbol{m}_1^T \boldsymbol{J} \boldsymbol{m}_2 \neq 0.$$

We can assume $\boldsymbol{m}_1^T \boldsymbol{J} \boldsymbol{m}_2 = 1$. Then,

$$\tilde{V} = \mathrm{Span}\left\{ \text{columns of } \begin{bmatrix} 1 & \mathbf{0} & 0 \\ \boldsymbol{m}_1 & \boldsymbol{M} & \boldsymbol{m}_2 \\ 0 & \mathbf{0} & 1 \end{bmatrix} \right\} \in \mathrm{Gr}^C(d, 2n).$$

Clearly, using a series of elementary row transformations (t-th row)-λ(1-st row), (t-th row)-λ(2n-th row), $2 \le t \le 2n-1$, $\lambda \in k$, we can obtain

$$\begin{bmatrix} 1 & \mathbf{0} & 0 \\ \boldsymbol{m}_1 & \boldsymbol{M} & \boldsymbol{m}_2 \\ 0 & \mathbf{0} & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \mathbf{0} & 0 \\ \mathbf{0} & \boldsymbol{M} & \mathbf{0} \\ 0 & \mathbf{0} & 1 \end{bmatrix}.$$

Furthermore, recall that in

$$\sum_{\underline{i} \in \Lambda} c_{\underline{i}} p_{\underline{i}},$$

$\forall \underline{i} \in \Lambda$, both $1, 2n \in \underline{i}$. Thus,

$$\sum_{\underline{i} \in \Lambda} c_{\underline{i}} p_{\underline{i}}\big(\varphi_3(V)\big) = \sum_{\underline{i} \in \Lambda} c_{\underline{i}} p_{\underline{i}}(\tilde{V}) = 0.$$

We obtain $\varphi_3^*(\sum_{\underline{i}\in\Lambda} c_{\underline{i}} p_{\underline{i}})|_{\mathrm{Gr}^C(d-2,2(n-1))} = 0$; then, by the induction assumption

$$\varphi_3^*\left(\sum_{\underline{i}\in\Lambda} c_{\underline{i}} p_{\underline{i}}\right) = \sum_{\underline{i''}\in I_{d-2,2(n-1)}} c_{\underline{i''}} E_{\underline{i''}}.$$

One preimage of $\sum_{\underline{i''}\in I_{d-2,2(n-1)}} c_{\underline{i''}} E_{\underline{i''}}$ under φ_3^* is $\sum_{\underline{i''}\in I_{d-2,2(n-1)}} c_{\underline{i''}} E_{\underline{i''+1}\cup\{1,2n\}}$. Additionally, $\ker\varphi_3^*$ is spanned by

$$\left\{p_{\underline{i}}, \underline{i}\in I_{d,2n} \,|\, 1\notin \underline{i} \ \text{ or } \ 2n\notin \underline{i}\right\}.$$

Comparing the subscripts, we have

$$\sum_{\underline{i}\in\Lambda} c_{\underline{i}} p_{\underline{i}} = \sum_{\underline{i''}\in I_{d-2,2(n-1)}} c_{\underline{i''}} E_{\underline{i''+1}\cup\{1,2n\}}.$$

The proof is complete. □

Finally, combining Propositions 2.3 and 2.4, we obtain the main theorem.

Theorem 3.5. *On a Grassmannian variety, for $\underline{i}\in I_{d,2n}^{Sp}$, the defining ideal of $X^C(\underline{i})$ in $X^A(\underline{i})$ is generated by $\{E_{\underline{i'}} \,|\, \underline{i'}\in I_{d-2,2n}\}$. In other words, $X^C(\underline{i})$ is a scheme-theoretical intersection of $X^A(\underline{i})$ with hyperplanes $\{E_{\underline{i'}} \,|\, \underline{i'}\in I_{d-2,2n}\}$. If $\underline{j}\in I_{d,2n}^{Sp}$ such that $\underline{j}\le\underline{i}$, then locally, the defining ideal of $X^C(\underline{i})\cap A_{\underline{j}}$ in $X^A(\underline{i})\cap A_{\underline{j}}$ is generated by $\{E_{\underline{i'}}/p_{\underline{j}} \,|\, \underline{i'}\in I_{d-2,2n}$, $\underline{i'}\subset\underline{i}\}$.*

On a flag variety, for $\underline{\mathbf{w}}\in\Delta_{d,2n}^{Sp}$, the defining ideal of $X^C(\underline{\mathbf{w}})$ in $X^A(\underline{\mathbf{w}})$ is generated by $\cup_{d=2}^n\{E_{\underline{i'}} \,|\, \underline{i'}\in I_{d-2,2n}\}$. In other words, $X^C(\underline{\mathbf{w}})$ is a scheme-theoretical intersection of $X^A(\underline{\mathbf{w}})$ with hyperplanes $\cup_{d=2}^n\{E_{\underline{i'}} \,|\, \underline{i'}\in I_{d-2,2n}\}$. If $\underline{u}\in\Delta_{d,2n}^{Sp}$ such that $\underline{u}\le\underline{\mathbf{w}}$, then locally, the defining ideal of $X^C(\underline{\mathbf{w}})\cap O_{\underline{u}}$ in $X^A(\underline{\mathbf{w}})\cap O_{\underline{u}}$ is generated by $\{E_{\underline{i'}}/p_{\underline{u}^{(n)}} \,|\, \underline{i'}\in I_{n-2,2n}, \underline{i'}\subset\underline{u}^{(n)}\}$.

4 Number of required defining equations in Schubert varieties

In this section, we discuss the number of defining equations required to obtain $X^C(\underline{i})$ from $X^A(\underline{i})\subset\mathrm{Gr}(d,2n)$. By Lemma 3.1, we can see that many of $\{E_{\underline{i'}} \,|\, \underline{i'}\in I_{d-2,2n}\}$ will be identically zero on $X^A(\underline{i})$. Moreover, from Theorem 3.3, we have $E_{\underline{i'}}|_{X^A(\underline{i})} = 0$ if and only if $\underline{i'}\cup\{t,2n+1-t\}\le\underline{i}$ for all $1\le t\le n$, $t\notin\underline{i'}$. Our motivation is to exactly count the number of such equations. Of course not only the situation of Grassmannian but also one of flag variety should be discussed, but from Lemma 3.1 and the following Lemma 4.1, it is sufficient to work on Grassmannian cases. We do this locally, that is, on every affine open $A_{\underline{j}}$.

Now, for $\underline{j} \le \underline{i} \in I^{Sp}_{d,2n}$, let $N_{\underline{j},\underline{i}}$ be the number

$$\#\Big\{1 \le s < t \le d \,|\, E_{\underline{j}} \backslash \{j_s, j_t\}|_{X^A(\underline{i})} \ne 0\Big\}.$$

Recall that

$$\begin{aligned}&W(\mathrm{Sp}_{2n}) \subset W(\mathrm{SL}_{2n}) = S_{2n},\\ &W(\mathrm{Sp}_{2n}) = \{w = (w_1 w_2 \cdots w_{2n}) \in S_{2n} \,|\, w_t + w_{2n+1-t} = 2n+1, \forall 1 \le t \le n\}.\end{aligned}$$

For $w \in W(\mathrm{SL}_{2n}) = S_{2n}$ written on one-line, the length is $l^A(w) = \tau(w)$, i.e., the inversion number, and for $w \in W(\mathrm{Sp}_{2n}), l^C(w) = (\tau(w)+m)/2$, where $m = \#\{1 \le t \le d \mid w_t > n\}$. We have the identifications $I_{d,2n} = W(\mathrm{SL}_{2n})^{\min}_{P^A_d}$ and $I^{Sp}_{d,2n} = W(\mathrm{Sp}_{2n})^{\min}_{P^C_d}$. However, these two identifications work in different ways; they are, respectively,

$$\begin{aligned}&\underline{i} = (i_1, i_2, \cdots, i_d) \;\mapsto\; \underline{i}^A = (i_1, i_2, \cdots, i_d, i_{d+1}, \cdots, i_{2n}) \in W(\mathrm{SL}_{2n}),\\ &\text{i.e.} \quad i_{d+1} < \cdots < i_{2n}\end{aligned}$$

and

$$\begin{aligned}&\underline{i} = (i_1, i_2, \cdots, i_d) \;\mapsto\; \underline{i}^C = (i_1, i_2, \cdots, i_d, i_{d+1}, \cdots, i_n, i_{n+1}, \cdots, i_{2n}) \in W(\mathrm{Sp}_{2n}),\\ &\text{i.e.} \quad i_{d+1} < \cdots < i_n, \quad \forall 1 \le t \le n, \quad i_t + i_{2n+1-t} = 2n+1.\end{aligned}$$

If $d = n$, then $\underline{i}^A = \underline{i}^C$, but in general, they are distinct.

We denote $l(\underline{i}, A) = l^A(\underline{i}^A)$ and $l(\underline{i}, C) = l^C(\underline{i}^C), \forall \underline{i} \in I^{Sp}_{d,2n}$.

Lemma 4.1 ([17, p.119]). *For* $\forall \underline{i} \in I^{Sp}_{d,2n}$,

$$\dim X^A(\underline{i}) = l(\underline{i}, A), \quad \dim X^C(\underline{i}) = l(\underline{i}, C).$$

For $\underline{\mathbf{w}} = (w_1, w_2, \cdots, w_n) \in \Delta^{Sp}_{n,2n}$,

$$\begin{aligned}\dim X^A(\underline{\mathbf{w}}) &= l\big(\underline{\mathbf{w}}^{(n)}, A\big) + \tau(w_1, w_2, \cdots, w_n),\\ \dim X^C(\underline{\mathbf{w}}) &= l\big(\underline{\mathbf{w}}^{(n)}, C\big) + \tau(w_1, w_2, \cdots, w_n).\end{aligned}$$

Example 4.1. Take $d=3, n=4$ and $\underline{i}=(137)\in I^{Sp}_{3,8}$

$$\underline{i}^A=(13724568) \quad \text{and} \quad \underline{i}^C=(13745268).$$

Thus,

$$\dim X^A\big((137)\big)=5, \quad \dim X^C\big((137)\big)=(7+1)/2=4.$$

Now, let us consider $N_{(137),(137)}$ and $N_{(123),(137)}$

$$\begin{aligned} E_1&=\pm p_{127}\pm p_{136}\pm p_{145},\\ E_3&=\pm p_{138}\pm p_{237}\pm p_{345},\\ E_7&=\pm p_{178}\pm p_{367}\pm p_{457}. \end{aligned}$$

By Lemma 3.1 and Theorem 3.3, we see that $N_{(137),(137)}=1$

$$\begin{aligned} E_1&=\pm p_{127}\pm p_{136}\pm p_{145},\\ E_2&=\pm p_{128}\pm p_{236}\pm p_{245},\\ E_3&=\pm p_{138}\pm p_{237}\pm p_{345}. \end{aligned}$$

So $N_{(123),(137)}=2$. Note that

$$N_{(137),(137)}=\dim X^A\big((137)\big)-\dim X^C\big((137)\big),$$

but

$$N_{(123),(137)}\neq\dim X^A\big((137)\big)-\dim X^C\big((137)\big).$$

Proposition 4.1. *$X^C(\underline{i})=X^A(\underline{i})\cap \mathrm{Gr}^C(d,2n)$, and in affine open $A_{\underline{i}}$, $N_{\underline{i},\underline{i}}=l(\underline{i},A)-l(\underline{i},C)$. Particularly, $X^C(\underline{i})\cap A_{\underline{i}}$ is a complete intersection of $X^A(\underline{i})\cap A_{\underline{i}}$.*

The proof of Proposition 4.1 relies on the following idea of exclusion.

Lemma 4.2. *For $\underline{i}\in I_{d,2n}$, $1\le s<t\le d$, if $i_s+i_t\le 2n$, then $E_{\underline{i}\backslash\{i_s,i_t\}}|_{X^A(\underline{i})}=0$.*

Proof. $E_{\underline{i}\backslash\{i_s,i_t\}}=\sum_{r=1}^n\pm p_{\underline{i}\backslash\{i_s,i_t\}\cup\{r,2n+1-r\}}$, and for every $\underline{j}=\underline{i}\backslash\{i_s,i_t\}\cup\{r,2n+1-r\}$, there must be $\underline{j}\not\le\underline{i}$ because

$$\sum_{k=1}^d j_k=\sum_{k=1}^d i_k-i_s-i_t+r+(2n+1-r)\ge\sum_{k=1}^d i_k+1.$$

By Lemma 3.1, $E_{\underline{i}\backslash\{i_s,i_t\}}|_{X^A(\underline{i})}=0$ holds. $\square$

Then, the number of equations $\#\{E_{\underline{i}\backslash\{i_s,i_t\}},1\le s<t\le d\mid E_{\underline{i}\backslash\{i_s,i_t\}}|_{X^A(\underline{i})}\neq 0\}$ is at most $\#\{1\le s<t\le d\mid i_s+i_t>2n\}$. Next, we show that this number is exactly $l(\underline{i},A)-l(\underline{i},C),\forall\underline{i}\in I^{Sp}_{d,2n}$.

Proposition 4.2. *For* $\underline{i}\in I^{Sp}_{d,2n}$, $\#\{1\le s<t\le d\,|\,i_s+i_t>2n\}=l(\underline{i},A)-l(\underline{i},C)$.

Proof. Fix $m=\#\{1\le k\le d\,|\,i_k>n\}$, use induction on $\sum_{k=1}^d i_k$. Note that for a fixed m, the minimal possible value of $\sum_{k=1}^d i_k$ is $1+2+\cdots+(d-m)+(n+1)+\cdots+(n+m)$. At this time,

$$\underline{i}=(1,2,\cdots,d-m,n+1,\cdots,n+m).$$

Its shortest representative in $W(\mathrm{Sp}_{2n})$ is

$$\underline{i}^C=(I,II,III,IV,V,VI).$$

We partition this expression into six parts

$$\begin{aligned}
&I=(1,2,\cdots,d-m),\\
&II=(n+1,n+2,\dots,n+m),\\
&III=(d-m+1,d-m+2,\dots,n-m),\\
&IV=(n-m+1,\cdots,2n+m-d),\\
&V=(n-m+1,n-m+2,\dots,n),\\
&VI=(2n+m-d+1,2n+m-d+2,\cdots,2n).
\end{aligned}$$

Then,

$$\begin{aligned}
&\tau\left(\underline{i}^C\right)=\tau(II,III)+\tau(II,V)+\tau(IV,V),\\
&\tau\left(\underline{i}^A\right)=\tau(II,III)+\tau(II,V).
\end{aligned}$$

Thus,

$$\begin{aligned}
l(\underline{i},A)-l(\underline{i},C)&=\frac{\tau(II,III)+\tau(II,V)-\tau(IV,V)-m}{2}\\
&=\frac{m(n-d)+m^2-m(n-d)-m}{2}\\
&=\frac{m^2-m}{2}.
\end{aligned}$$

Additionally,

$$\#\{1\le s<t\le d\,|\,i_s+i_t>2n\}=\frac{m(m-1)}{2}=l(\underline{i},A)-l(\underline{i},C).$$

Now, for the minimal $\sum_{k=1}^d i_k$, our proposition is proved.

If $\sum_{k=1}^d i_k$ is greater than the minimal case, there must be some $i_k\neq 1,n+1$ such that $i_k-1\notin\underline{i}$; we can choose i_r to be the minimal such value. Then,

Case 1. $2n+1-(i_r-1)\notin \underline{i}$.

Let $\underline{j}=(\underline{i}\backslash i_r)\cup(i_r-1)\in I^{Sp}_{d,2n}$. Then,

$$l(\underline{j},A)=l(\underline{i},A)-1,\quad l(\underline{j},C)=l(\underline{i},C)-1.$$

Additionally,

$$\#\{1\le s<t\le d\,|\,j_s+j_t>2n\}=\#\{1\le s<t\le d\,|\,i_s+i_t>2n\},$$

because for those i_k such that $i_k+i_r>2n$, we have $i_k+i_r\neq 2n+1$, $i_k+i_r>2n+1$, which implies $i_k+(i_r-1)>2n$ as well. Note that $\sum_{k=1}^d j_k=\sum_{k=1}^d i_k-1$, so we can use induction assuming that

$$l(\underline{j},A)-l(\underline{j},C)=\#\{1\le s<t\le d\,|\,j_s+j_t>2n\}.$$

Therefore,

$$l(\underline{i},A)-l(\underline{i},C)=\#\{1\le s<t\le d\,|\,i_s+i_t>2n\}$$

holds.

Case 2. $2n+1-(i_r-1)\in \underline{i}$.

Let $\underline{j}=(\underline{i}\backslash\{i_r,2n+1-i_r\})\cup\{i_r-1,2n+1-i_r\}$. Then,

$$l(\underline{j},A)=l(\underline{i},A)-2,\quad l(\underline{j},C)=l(\underline{i},C)-1,$$

and

$$\#\{1\le s<t\le d\,|\,j_s+j_t>2n\}=\#\{1\le s<t\le d\,|\,i_s+i_t>2n\},$$

since the only removed such pair is

$$i_r,2n+1-i_r,$$

and the new pair $(i_r-1)+(2n+1-i_r)=2n\not>2n$.

$\sum_{k=1}^d j_k=\sum_{k=1}^d i_k-2$, by means of the induction assumption, we can obtain

$$l(\underline{j},A)-l(\underline{j},C)=\#\{1\le s<t\le d\,|\,j_s+j_t>2n\}.$$

Therefore,

$$l(\underline{i},A)-l(\underline{i},C)=\#\{1\le s<t\le d\,|\,i_s+i_t>2n\}.$$

The proof is complete. $\square$

Thus, Proposition 4.1 is also proved.

Lemma 4.3. *For $\underline{i}\in I^{Sp}_{d,2n}$ and $\underline{i'},\underline{j'}\in I^{Sp}_{d-2,2n}$ such that $\underline{j'}\leq\underline{i'}$, if $E_{\underline{i'}}|_{X^A(\underline{i})}\neq 0$, then $E_{\underline{j'}}|_{X^A(\underline{i})}\neq 0$.*

Proof. Without loss of generality, we can assume $l(\underline{j'},C)=l(\underline{i'},C)-1$, i.e. $\underline{j'}=\underline{i'}\backslash\{s\}\cup\{s-1\}$ or $\underline{j'}=\underline{i'}\backslash\{2n+2-s\}\cup\{2n+1-s\}$ or $\underline{j'}=\underline{i'}\backslash\{s,2n+2-s\}\cup\{s-1,2n+1-s\}$ for some $2\leq s\leq n$.

$E_{\underline{i'}}|_{X^A(\underline{i})}\neq 0$ implies that there exist $1\leq t\leq n$ satisfying $t,2n+1-t\notin\underline{i'}$ and $\underline{i'}\cup\{t,2n+1-t\}\leq\underline{i}$. If $t\neq s-1$, there must be $\underline{j'}\cap\{t,2n+1-t\}\neq\emptyset$. Furthermore, t (also, $2n+1-t$) appears at the same position in $\underline{j'}\cup\{t,2n+1-t\}$ and in $\underline{i'}\cup\{t,2n+1-t\}$. Easily

$$\underline{j'}\cup\{t,2n+1-t\}\leq\underline{i'}\cup\{t,2n+1-t\}\leq\underline{i}.$$

Then, $E_{\underline{j'}}|_{X^A(\underline{i})}\neq 0$.

If $t=s-1$, then there is only the possibility $\underline{j'}=\underline{i'}\backslash\{s\}\cup\{s-1\}$ or $\underline{j'}=\underline{i'}\backslash\{2n+2-s\}\cup\{2n+1-s\}$. At this time, by means of a straightforward comparison of

$$\begin{aligned}&\underline{j'}\cup\{t+1,2n-t\}, && \text{if} \quad \underline{j'}=\underline{i'}\backslash\{s\}\cup\{s-1\},\\ &\underline{j'}\cup\{t-1,2n+2-t\}, && \text{if} \quad \underline{j'}=\underline{i'}\backslash\{2n+2-s\}\cup\{2n+1-s\}\end{aligned}$$

with $\underline{i'}\cup\{t,2n+1-t\}$, we can see there must be $E_{\underline{j'}}|_{X^A(\underline{i})}\neq 0$. $\square$

Corollary 4.1. *Let $\underline{i}\in I^{Sp}_{d,2n}$. If $\underline{l}\leq\underline{j}\in I^{Sp}_{d,2n}$, then the number*

$$N_{\underline{l},\underline{i}}\geq N_{\underline{j},\underline{i}}.$$

Specifically,

$$N_{\underline{i},\underline{i}}=\#\{1\leq s<t\leq d\,|\,i_s+i_t>2n\}=l(\underline{i},A)-l(\underline{i},C)$$

and for all $\underline{j}\leq\underline{i}$,

$$N_{id,\underline{i}}\geq N_{\underline{j},\underline{i}}.$$

Proposition 4.3. *For $\underline{i}\in I^{Sp}_{d,2n}$ (resp. $\underline{\mathbf{w}}\in\Delta^{Sp}_{n,2n}$), $X^C(\underline{i})$ (resp. $X^C(\underline{\mathbf{w}})$) is a local complete intersection in $X^A(\underline{i})$ (resp. $X^A(\underline{\mathbf{w}})$) if and only if*

$$N_{id,\underline{i}}=N_{\underline{i},\underline{i}}$$

(resp. $N_{id,\underline{\mathbf{w}}^{(n)}}=N_{\underline{\mathbf{w}}^{(n)},\underline{\mathbf{w}}^{(n)}}$).

Proof. We prove this statement in only the Grassmannian case. The Bruhat decomposition of symplectic $\mathrm{Gr}^C(d,2n)=\cup_{\underline{j}\in I^{Sp}_{d,2n}}B^C\cdot e_{\underline{j}}$ implies that $\mathrm{Gr}^C(d,2n)$ can be covered by affine open $A_{\underline{j}} \cap \mathrm{Gr}^C(d,2n)$, $\underline{j} \in I^{Sp}_{d,2n}$. Thus, by Proposition 4.2 and Corollary 4.1, the sufficiency is obtained. To see the necessity, if

$$N_{id,\underline{i}} > N_{\underline{i},\underline{i}},$$

let m_{id} be the maximal ideal of the local ring of e_{id} in $X^A(\underline{i}) \cap A_{id}$. Then, the residues of

$$\left\{ \frac{E_{id\setminus\{s,t\}}}{p_{id}}, 1 \leq s < t \leq d \,|\, E_{id\setminus\{s,t\}}|_{X^A(\underline{i})} \neq 0 \right\}$$

are linearly independent in m_{id}/m_{id}^2. That implies that m_{id} cannot be generated by less than $N_{id,\underline{i}}$ elements. Thus, $X^C(\underline{i})$ cannot be a complete intersection of $X^A(\underline{i})$ in any open neighborhood. □

Proposition 4.4. *Let $\underline{i} \in I^{Sp}_{d,2n}$ with $N_{\underline{i},\underline{i}} > 0$. If $r_1 = \min\{1 \leq a \leq d \,|\, i_a \geq a+1\}$, $r_2 = \min\{1 \leq a \leq d \,|\, i_a \geq a+2\}$, $q = 2n+1-i_d$, then the number*

$$N_{id,\underline{i}} = \binom{d-(r_1-1)}{2} - \big(\min\{r_2,q\} - r_1\big).$$

Proof. By the remark after Theorem 3.4, we need to prove this statement only for $r_1 = 1$, i.e., $i_1 \geq 2$. Denote $l = \min\{r_2,q\}$. First, let us consider $l = 1$, i.e., $i_1 \geq 3$ or $i_d = 2n$.

In the case $i_1 \geq 3$, since $N_{\underline{i},\underline{i}} > 0$, for some $1 \leq s < t \leq d$, we have $E_{\underline{i}\setminus\{i_s,i_t\}}|_{X^A(\underline{i})} \neq 0$. There must be $\underline{i}\setminus\{i_s,i_t\} \geq (3,4,\ldots,d)$, where $(3,4,\ldots,d)$ is the unique maximal (under Bruhat order) element among all the $id\setminus\{s',t'\}, 1 \leq s' < t' \leq d$. Thus, for any $id\setminus\{s',t'\}$, $1 \leq s' < t' \leq d$, $E_{id\setminus\{s',t'\}}|_{X^A(\underline{i})} \neq 0$ (Corollary 4.1). Then, $N_{\underline{i},\underline{i}} \geq \binom{d}{2}$, so it must be $\binom{d}{2}$.

In the case $i_d = 2n$, there must be $E_{\underline{i}\setminus\{i_1,i_d\}}|_{X^A(\underline{i}} \neq 0$. $i_2 > i_1 \geq 2$, so $\underline{i}\setminus\{i_1,i_d\} \geq (3,4,\ldots,d-2)$. Then, a similar reasoning reveals $N_{\underline{i},\underline{i}} = \binom{d}{2}$ as well.

Now, let $l > 1$. Then $i_1 = 2, i_2 = 3, \cdots, i_{l-1} = l$ and $i_d \neq 2n$. Consequently, there must be $i_d < 2n+1-l$, $l < 2n+1-i_d = q$; therefore, $l = r_2$, $i_l \geq l+2$.

Since $N_{\underline{i},\underline{i}} > 0$ and $i_1 = 2, i_2 = 3, \cdots, i_{l-1} = l$, $i_d < 2n+1-l$, for some $l \leq s < t \leq d$, we have $E_{\underline{i}\setminus\{i_s,i_t\}}|_{X^A(\underline{i})} \neq 0$. There must be $\underline{i}\setminus\{i_s,i_t\} \geq (2,3,\cdots,l,l+2,\cdots,d)$, where $(2,3,\cdots,l,l+2,\cdots,d)$ is the unique maximal (under Bruhat order) element in

$$\big\{\underline{i'} = id\setminus\{s',t'\}, 1 \leq s' < t' \leq d \,|\, i'_{l-1} \leq l\big\}.$$

This set is exactly

$$\big\{\underline{i'} = id\setminus\{s',t'\}, 1 \leq s' < t' \leq d\big\} \setminus \big\{id\setminus\{1,k\} \,|\, 2 \leq k \leq l\big\}.$$

By Corollary 4.1, for any $\underline{i'}=id\backslash\{s',t'\}$ with $i'_{l-1}\leq l$, we have $E_{id\backslash\{s',t'\}}|_{X^A(\underline{i})}\neq 0$. Additionally, if $\underline{i'}=id\backslash\{1,k\}, 2\leq k\leq l$, we can see that $\underline{i'}\cup\{a,2n+1-a\}\not\leq\underline{i}$ for any $1\leq a\leq n, a\notin\underline{i'}$, which implies $E_{\underline{i'}}|_{X^A(\underline{i})}=0$. Then, $N_{id,\underline{i}}=\binom{d}{2}-(l-1)$.

By replacing d,r_2, and q by $d-(r_1-1)$, $r_2-(r_1-1)$, and $q-(r_1-1)$, we obtain the general formula. $\square$

Remark 4.1. A special situation is $d=2$. If $N_{\underline{i},\underline{i}}>0$, then there must be $l=1$.

Theorem 4.1. *For $\underline{i}\in I^{Sp}_{d,2n}$ and $r_1=\min\{1\leq a\leq d\,|\,i_a\geq a+1\}$, the symplectic Schubert variety $X^C(\underline{i})$ is a local complete intersection in $X^A(\underline{i})$ if and only if $i_s+i_t>2n,\forall r_1\leq s<t\leq d$. At this time, if $X^A(\underline{i})$ is smooth, then $X^C(\underline{i})$ is a local complete intersection (intrinsically).*

Proof. Let r_1,r_2,q,l be as in the proposition above. We only discuss the case for $N_{\underline{i},\underline{i}}>0$ and $r_1=1$. If $l>1$ (then $d>2$), we have $i_1=2, i_2=3,\cdots,i_{l-1}=l$, $i_d<2n+1-l$. Thus, $N_{\underline{i},\underline{i}}=\#\{1\leq s<t\leq d\,|\,i_s+i_t>2n\}\leq\binom{d-(l-1)}{2}$. However, $\binom{d}{2}-(l-1)\leq\binom{d-(l-1)}{2}$ only occurs when $d\leq 2$. It follows that $X^C(\underline{i})$ cannot be a local complete intersection of $X^A(\underline{i})$ for $l>1$.

Thus, there must be $l=1$. At this time, $N_{id,\underline{i}}=\binom{d}{2}$. Therefore, if we want $N_{id,\underline{i}}=N_{\underline{i},\underline{i}}$, it is equivalent to $i_s+i_t>2n,\forall 1\leq s<t\leq d$. In general (not necessarily $r_1=1$), it is $i_s+i_t>2n,\forall r_1\leq s<t\leq d$. $\square$

Corollary 4.2. *For $\underline{\mathbf{w}}\in\Delta^{Sp}_{n,2n}$ and $r_1=\min\{1\leq a\leq n\,|\,a\notin\underline{\mathbf{w}}\}$, the symplectic Schubert variety $X^C(\underline{i})$ is a local complete intersection in $X^A(\underline{i})$ if and only if $\underline{\mathbf{w}}^{(n)}=(1,2,\cdots,r_1-1,n,n+2,\cdots,2n+1-r_1)$. Moreover, it is equivalent to stating that the projection image $pr_n(X^A(\underline{\mathbf{w}}))=X^A(\underline{\mathbf{w}}^{(n)})\subset\mathrm{Gr}(n,2n)$ is either a smooth Schubert variety or of codimension 1 in a smooth Schubert variety in $\mathrm{Gr}(n,2n)$. At this time, if $X^A(\underline{\mathbf{w}})$ is smooth, then $X^C(\underline{\mathbf{w}})$ is a local complete intersection (intrinsically).*

Proof. See [15, (5.3)] for the involved criteria for smoothness of Schubert varieties in $\mathrm{Gr}(n,2n)$. Clearly, $pr_n(X^A(\underline{\mathbf{w}}))=X^A(\underline{\mathbf{w}}^{(n)})$ satisfies the condition in Theorem 4.1 if and only if it is smooth or of codimension 1 in a smooth Schubert variety. $\square$

5 Symplectic conditions on tangent space

Except for the well-studied situation of flag varieties, existing research on the singularities of Schubert varieties concentrates on the case where this Schubert variety lies in a minuscule or cominuscule G/P, where G is a classical linear algebraic group and P is the minuscule or cominuscule maximal parabolic subgroup of G. In SL_{2n}, every maximal parabolic subgroup $P_d^A, 1\le d\le 2n$ is minuscule and cominuscule [15]. However, in Sp_{2n}, there are only P_1^C and P_n^C. This leaves a gap in the research on $\mathrm{Gr}^C(d,2n), 1<d<n$. For $\mathrm{Gr}^C(d,2n)$ (and even $\mathrm{Fl}^C_{2n}(1,2,\dots,n)$), there is an interesting results stating that $X^C(\underline{i})$ (resp. $X^C(\underline{\mathbf{w}})$) must be smooth if $X^A(\underline{i})$ is already smooth [15] (resp. its corresponding Schubert variety in $\mathrm{Fl}_{2n}(1,2,\dots,2n)$ is smooth [14]). In this section, we compute the codimension of tangent spaces and note that a type C Schubert variety in $\mathrm{Gr}^C(d,2n)$ is not necessarily smooth even if its type A Schubert variety is smooth. Additionally, the condition making such Schubert variety smooth is given.

First, a Schubert variety $X(w)\subset G/P$ is smooth if and only it is smooth at the point e_{id} where $id\in W(G)_P^{\min}$ is the identity in the Weyl group. That is because the singular points of an algebraic variety form a closed subvariety and because the fact that one point in a Schubert variety is singular implies that all points in the same B-orbit with this point are singular. Thus, to discuss the smoothness of a Schubert variety, we need to study only its tangent space at e_{id}.

For $\underline{i}\in I^{Sp}_{d,2n}$, we want to find the codimension of $T_{e_{id}}(X^C(\underline{i}))$ in $T_{e_{id}}(X^A(\underline{i}))$. Since $X^C(\underline{i})=X^A(\underline{i})\cap\mathrm{Gr}^C(d,2n)$, (without confusion we write $E_{\underline{i}}$ instead of $E_{\underline{i}}/p_{id}$)

$$T_{e_{id}}(X^C(\underline{i}))=T_{e_{id}}(X^A(\underline{i}))\cap\Big(\bigcap_{\underline{i}'\in I_{d-2,2n},\underline{i}'\subset id}T_{e_{id}}(E_{\underline{i}'})\Big).$$

To determine the codimension, the first problem is to find those $\underline{i}'\subset id$ satisfying $T_{e_{id}}(X^A(\underline{i}))\not\subset T_{e_{id}}(E_{\underline{i}'})$. Note that identifying $T_{e_{id}}(\mathrm{GL}_{2n}/P_d^A)$ with the affine space A_{id} then the basis of $T_{e_{id}}(X^A(\underline{i}))$ is given by (cf. [15, Theorem 2.4])

$$\{x_{st},1\le t\le d<s\le 2n\,|\,(s\leftrightarrow t)\le\underline{i}\},$$

where $(s\leftrightarrow t)$ is the transposition exchanging s and t, and "$\le$" is under the Bruhat order.

In A_{id},

$$E_{id\setminus\{s,t\}}=\pm(x_{2n+1-s,t}-x_{2n+1-t,s}+\text{deg-2 terms}).$$

Thus, $T_{e_{id}}E_{id\setminus\{s,t\}}$ is the hyperplane $x_{2n+1-s,t}-x_{2n+1-t,s}$. Then $T_{e_{id}}(X^A(\underline{i}))\subset T_{e_{id}}(E_{\underline{i}'})$ if and only if both the transpositions $(2n+1-s\leftrightarrow t)\not\le\underline{i}$ and $(2n+1-t\leftrightarrow s)\not\le\underline{i}$. That is, $T_{e_{id}}(X^A(\underline{i}))\not\subset T_{e_{id}}(E_{\underline{i}'})$ if and only if one of the following statements hold:

$$(2n+1-t\leftrightarrow s)=(1,2,\cdots,s-1,s+1,\cdots,t-1,t,t+1,\cdots,d,2n+1-t)\le\underline{i}$$

or

$$(2n+1-s\leftrightarrow t)=(1,2,\cdots,s-1,s,s+1,\cdots,t-1,t+1,\cdots,d,2n+1-s)\le\underline{i}.$$

At this time, we say such $E_{id\setminus\{s,t\}}$ contributes one codimension. Through our discussion above it is easy to see the codimension of $T_{e_{id}}(X^C(\underline{i}))$ in $T_{e_{id}}(X^A(\underline{i}))$ is exactly the number of $E_{id\setminus\{s,t\}}$ which contributes some codimension.

Theorem 5.1. *Let* $\underline{i}\in I_{d,2n}^{Sp}$ *and* $r=\min\{1\le k\le d\,|\,i_k\ge k+1\}$, $q=2n+1-i_d\ge r$. *Then, the codimension of* $T_{e_{id}}(X^C(\underline{i}))$ *in* $T_{e_{id}}(X^A(\underline{i}))$ *is*

$$\begin{cases}\dfrac{(d-q+1)(d+q-2r)}{2}=\dbinom{d-r+1}{2}-\dbinom{q-r}{2}, & \text{if} \quad q\le d,\\ 0, & \text{if} \quad q>d.\end{cases}$$

Proof. Consider $1\le s<t\le d$.

Case 1. $s<t<r$. At this time,

$$\begin{array}{l|c|c|c|ll}
(1,2,\cdots,s-1, & s+1, & \cdots,t-1, & t, & t+1, & \cdots,d,2n+1-t)\\
(1,2,\cdots,s-1, & s, & s+1,\cdots,t-1, & t+1, & & \cdots,d,2n+1-s)\\
(1,2,\cdots,s-1, & s, & s+1,\cdots,t-1, & t, & \cdots,r-1, & \cdots,2n+1-q)
\end{array}$$

(the three lines are $(2n+1-t\leftrightarrow s),(2n+1-s\leftrightarrow t)$, and $\underline{i}$ in order, similar for Cases 2 and 3). Since $s+1>s$ and $t+1>1$ at the marked position, $E_{id\setminus\{s,t\}}$ contributes no codimension.

Case 2. $s<r\le t$.

$$\begin{array}{llll|l}
(1,2,\cdots,s-1, & \cdots & , & \cdots,d, & 2n+1-t)\\
(1,2,\cdots,s-1, & \cdots & , & \cdots,d, & 2n+1-s)\\
(1,2,\cdots,s-1, & \cdots & , & \cdots, & 2n+1-q).
\end{array}$$

Note that $2n+1-s>2n+1-r\ge 2n+1-q$, $E_{id\setminus\{s,t\}}$ contributes no codimension.

Case 3. $r\le s<t$.

$$\begin{array}{lll|lllll|l}
(1, & 2, & \cdots,s-1, & s+1, & \cdots,t-1, & t, & t+1, & \cdots,d, & 2n+1-t)\\
(1, & 2, & \cdots,s-1, & s, & s+1,\cdots,t-1, & t+1, & & \cdots,d, & 2n+1-s)\\
(1, & \cdots, & r-1,\cdots, & \cdots, & & \cdots, & & \cdots, & 2n+1-q).
\end{array}$$

For those s,t satisfying $2n+1-s>2n+1-t>2n+1-q$, $E_{id\setminus\{s,t\}}$ contributes no codimension. Additionally, for $r\le s,q\le t$, every equation $E_{id\setminus\{s,t\}}$ contributes codimension 1.

Thus, for $q>d$, the codimension is zero. For $q\le d$, the codimension of $T_{e_{id}}(X^C(\underline{i}))$ in $T_{e_{id}}(X^A(\underline{i}))$ is

$$\begin{aligned}\#\{s<t\,|\,r\le s,q\le t\}&=(d-q+1)(q-r)+\frac{(d-q+1)(d-q)}{2}\\&=\frac{(d-q+1)(d+q-2r)}{2}.\end{aligned}$$

The proof is complete. □

This result also reveals an interesting fact: the codimension of the tangent space $T_{e_{id}}(X^C(\underline{i}))$ in $T_{e_{id}}(X^A(\underline{i}))$ depends only on the indexes d,q,r defined above rather than more specific form of $\underline{i}$. By means of this theorem and the criterion for smoothness of $X^A(\underline{i})\subset \mathrm{Gr}(d,2n)$, $1<d<n$ from Lakshmibai and Weyman [15, (5.3)], we can obtain the following result.

Corollary 5.1. *For $1<d<n$, $\underline{i}\in I^{Sp}_{d,2n}$ satisfying that $X^A(\underline{i})$ is smooth, then $X^C(\underline{i})$ is smooth if and only if one of the following conditions holds: (q,r as defined in Theorem 5.1)*

(1) $q>n$ (then trivially $X^C(\underline{i})=X^A(\underline{i})$),

(2) $q=r$,

(3) $q=r+1$.

Proof. By [15, (5.3)] the smoothness of $X^A(\underline{i})$ implies $\underline{i}=(1,2,\cdots,r-1,t,t+1,\cdots,t+d-r)$, where t is some integer greater than r. Since $\underline{i}\in I^{Sp}_{d,2n}$, there must be $t\ge n$ or $i_d=t+d-r\le n$.

If $t\ge n$, by Proposition 4.2 we have $\dim X^A(\underline{i})-\dim X^C(\underline{i})=\binom{d-r+1}{2}$. Therefore, $X^C(\underline{i})$ is smooth if and only if $q=r$ or $q=r+1$ by Theorem 5.1. If $i_d\le n$, the case is trivially $X^C(\underline{i})=X^A(\underline{i})$. □

In general, we have the following criterion immediately.

Corollary 5.2. *For $1\le d\le n$, $\underline{i}\in I^{Sp}_{d,2n}$, q,r as defined in Theorem 5.1, the Schubert variety $X^C(\underline{i})$ is smooth if and only if*

$$\dim T_{e_{id}}\big(X^A(\underline{i})\big)-\dim X^A(\underline{i})=\binom{d-r+1}{2}-\binom{q-r}{2}-\#\{1\le s<t\le d\,|\,i_s+i_t>2n\}.$$

Acknowledgments

We thank the editors and reviewers for their valuable time and helpful suggestions.

This work was supported by the National Natural Science Foundation of China (Grant No. 12271347).

References

[1] D. Anderson, T. Ikeda, M. Jeon, and R. Kawago, *Multiplicities of Schubert varieties in the symplectic flag variety*, Sémin. Lothar. Comb. 82B (2019), Paper No. 95, 12 pp.

[2] S. Billey and V. Lakshmibai, *Singular Loci of Schubert Varieties*, Birkhäuser Boston, 2000.

[3] G. Bobiński and G. Zwara, *Schubert varieties and representations of Dynkin quivers*, Colloq. Math. 94 (2002), 285–309.

[4] C. De Concini, *Symplectic standard tableaux*, Adv. Math. 34 (1979), 1–27.

[5] W. Fulton, *Representation Theory. A First Course*, Springer-Verlag, 1991.

[6] S. R. Ghorpade and K. N. Raghavan, *Hilbert functions of points on Schubert varieties in the symplectic Grassmannian*, Trans. Amer. Math. Soc. 358 (2006), 5401–5423.

[7] J. Hong, *Classification of smooth Schubert varieties in the symplectic Grassmannians*, J. Korean Math. Soc. 52 (2015), 1109–1122.

[8] R. Kinser and J. Rajchgot, *Type A quiver loci and Schubert varieties*, J. Commut. Algebra 7 (2015), 265–301.

[9] R. Kinser and J. Rajchgot, *Type D quiver representation varieties, double Grassmannians, and symmetric varieties*, Adv. Math. 376 (2021), Paper No. 107454, 44 pp.

[10] V. Lakshmibai, *Geometry of G/P - VII - The symplectic group and the involution σ*, J. Algebra 108 (1987), 403–434.

[11] V. Lakshmibai, *Singular loci of Schubert varieties for classical groups*, Bull. Amer. Math. Soc. 16 (1987), 83–90.

[12] V. Lakshmibai and J. Brown, *Flag Varieties*, Springer Singapore, 2018.

[13] V. Lakshmibai and K. N. Raghavan, *Symplectic Grassmannian*, in: Standard Monomial Theory: Invariant Theoretic Approach, Springer (2008), 55–69.

[14] V. Lakshmibai and M. Song, *A criterion for smoothness of Schubert varieties in* Sp_{2n}/B, J. Algebra 189 (1997), 332–352.

[15] V. Lakshmibai and J. Weyman, *Multiplicities of points on a Schubert variety in a minuscule G/P*, Adv. Math. 84 (1990), 179–208.

[16] J. C. Pacheco and F. Zaldivar, *On Lagrangian-Grassmannian codes*, Des. Codes Cryptogr. 60 (2011), 291–298.

[17] C. S. Seshadri, *Introduction to the Theory of Standard Monomials*, Springer Singapore, 2016.
[18] T. A. Springer, *Linear Algebraic Groups*, Birkhäuser Boston, 1998.