Anal. Theory Appl.
Vol. 28, No. 1 (2012), 27–37

THE HAUSDORFF MEASURE OF SIERPINSKI CARPETS BASING ON REGULAR PENTAGON*

Chaoyi Zeng, Dehui Yuan
(*Hanshan Normal University, China*)
and
Shaoyuan Xu**
(*Gannan Normal University, China)*

Received Jan. 19, 2011

Abstract. In this paper, we address the problem of exact computation of the Hausdorff measure of a class of Sierpinski carpets E− the self-similar sets generating in a unit regular pentagon on the plane. Under some conditions, we show the natural covering is the best one, and the Hausdorff measures of those sets are euqal to $|E|^s$, where $s = \dim_H E$.

Key words: *Sierpinski carpet, Hausdorff measure, upper convex density*

AMS (2010) subject classification: 28A78, 28A80

1 Introduction

The Hausdorff measure and dimension are the most important concepts in fractal geometry, and their computation is very difficult. Recently, in order to study deeply the Hausdorff measure, the reference [1] gave the notions “best covering" and “natural covering", and posed eight open problems and six conjectures on Hausdorff measure. Using the notion upper convex density of a class of self-similar sets, the reference [2] studied a class of self-similar sets-generating in a unit square on the plane, proved that the natural covering is the best one and the Hausdorff measures of those sets are euqal to $\sqrt{2}^s$. In this paper, we address the problem of the exact computation

* Partially supported by National Natural Science Foundation of China (No. 10961003).

Corresponding author.

of the Hausdorff measure of a class self similar sets-generating in a unit regular pentagon on the plane.

For convenience, we first present some notions that will be used in the rest part of the paper.

Definition 1. Suppose $E \subset \mathbf{R}^2$, $s \in \mathbf{R}$, $s \geq 0$ and $\delta > 0$, the Hausdorff measure of the set E is defined as

$$H^s(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : |U_i| \leqslant \delta, E \subset \bigcup U_i \right\}$$

where $\{U_i\}_{i=1}^{\infty}$ is arbitrary covering of the set E; and the Hausdorff dimension of E (denoted by $\dim_H E$) is defined as

$$\dim_H E = \sup\{s : H^s(E) = \infty\} = \inf\{s : H^s(E) = 0\}.$$

Definition 2. Let $\delta > 0$, $s \geq 0$, $E \subset R^2$, $x \in E$. Moreover, for a convex set U_x containing x, the upper convex density of E at x is defined as

$$\bar{D}_C^s(E,x) = \lim_{\delta \to 0} \sup_{0<|U_x|<\delta} \left\{ \frac{H^s(E \cap U_x)}{|U_x|^s} \right\}.$$

The properties of the upper convex density are discussed in the reference [5].

Definition 3. (See Fig. 1) Let E_0 be an unit regular pentagon $A_1A_2A_3A_4A_5$ on the plane R^2, E be the attractor generated by the iterated function system (IFS) $\{f_i | i = 1,2,3,4,5\}$, where

$$f_i(x) = \lambda_i x + b_i, 0 < \lambda_i < 1, i = 1,2,3,4,5, x = (x_1, x_2) \in E_0$$
$$b_1 = ((1-\lambda_1)\sin 18^\circ, 0)$$
$$b_2 = ((1-\lambda_2)(\sin 18^\circ + 1), 0)$$
$$b_3 = ((1-\lambda_3)(2\sin 18^\circ + 1), (1-\lambda_3)\cos 18^\circ)$$
$$b_4 = ((1-\lambda_4)\cos 18^\circ, (1-\lambda_4)(\sin 18^\circ + \cos 18^\circ))$$
$$b_5 = (0, (1-\lambda_5)\cos 18^\circ)$$

Then the self-similar set E is called a Sierpinski carpet generating in a unit regular pentagon, where $s = \dim_H E$ satisfies $\sum_{i=1}^{5} \lambda_i^s = 1$.

2 Two Lemmas

In this section, we present two lemmas which will be used in the proof of the main result of this paper.

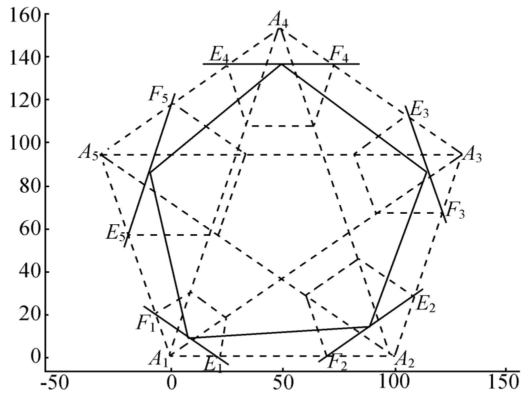

Fig. 1

From the definition 3, it is easy to see that

$$\bigcup_{i=1}^{5} f_i(E_0) \subset E_0 \qquad \text{and} \qquad \bigcup_{i=1}^{5} f_i(E) = E.$$

Let μ be the unique probability measure satisfying the self-similar relation and

$$\mu = \sum_{i=1}^{5} \lambda_i^s \mu f_i^{-1},$$

then E is the support of μ and μ is a mass distribution on E.

For $i = 1,2,3,4,5$, let E_iF_i be parallel to the opposite side of vertex A_i in an unit regular pentagon and intersect $f_i(E)$, let $d_i = \text{dist}(A_i, E_iF_i)$ be the distance between point A_i and line E_iF_i. Denote

$$t_i = \frac{d_i}{\cos 18^\circ},$$

if

$$0 < \lambda_i < \frac{7-2\sqrt{5}}{9}$$

and $0 < d_i < (\sqrt{5}+1)\lambda_i$, then the line E_iF_i doesn't intersect $f_j(E)$ for $i,j \in \{1,2,3,4,5\}$ and $i \neq j$. Assume that $\mu(t_i)$ is the measure of the triangle $\Delta A_iE_iF_i$. Moreover, we use the notations:

$$d(\mu,t_i) = \frac{\mu(t_i)}{t_i^s}, d_{\min}^{(i)} = \inf_{0<t_i\le 2\sin 54^\circ}\{d(\mu,t_i)\}, i = 1,2,3,4,5$$

$$M_1 = \{(i,j)|A_iA_j \text{ is the diagonal of } E_0, i,j = 1,2,3,4,5\}$$

and

$$M_2=\left\{(i,j)|A_iA_j \text{ is the side of } E_0, i,j=1,2,3,4,5\right\}.$$

That is

$$M_1=\{(1,3),(3,1),(1,4),(4,1),(2,4),(4,2),(2,5),(5,2),(3,5),(5,3)\}$$

and

$$M_2=\left\{(1,2),(2,1),(2,3),(3,2),(3,4),(4,3),(4,5),(5,4),(1,5),(5,1)\right\}.$$

Lemma 4. *Let*

$$0<\lambda_i<\frac{7-2\sqrt{5}}{9},\quad 0<s<1,\quad i=1,2,3,4,5.$$

Assume that K is a nonnegative integer. Then $d(\mu,t_i)$ attains its infimum $d_{\min}^{(i)}$ only at the following values of t_i:

(1) $t_i=\lambda_i^K\dfrac{1-\lambda_p}{2\sin 18^\circ},(i,p)\in M_1$, *or*

(2) $t_i=\lambda_i^K\cdot\min\limits_{i\neq q,p\neq q}\{2\sin 18^\circ(1-\lambda_q)\},(i,q)\in M_2$.

Proof. Since E is self-similar, then we need only prove that the result is true when $2\lambda_1\sin 54^\circ<t_1\le 2\sin 54^\circ$ for $i=1$. Denote

$$r_1=\lambda_1\sin 54^\circ, r_2=\min\{2(1-\lambda_2)\sin 18^\circ,2(1-\lambda_5)\sin 18^\circ\},$$
$$r_3=2\sin 54^\circ-\min\{\mathbf{2}(1-\lambda_2)\sin 18^\circ,2(1-\lambda_5)\sin 18^\circ\},$$
$$r_3=2\sin 54^\circ-\max\{2\lambda_3\sin 54^\circ,2\lambda_4\sin 54^\circ\}.$$

Case 1. $d(\mu,t_1)$ attains its infimum $d_{\min}^{(1)}$ at the interval $(r_2,r_3]$. In this case, the line E_1F_1 intersects $f_2(E)$ or $f_5(E)$. Therefore

$$\begin{aligned}
d(\mu,t_1)&=\frac{\mu(t_1)}{t_1^s}\\
&=\frac{\lambda_1^s+\mu(t_1-2(1-\lambda_2)\sin 18^\circ)+\mu(t_1-2(1-\lambda_5)\sin 18^\circ)}{(r_2+t_1-r_2)^s}\\
&\ge\frac{\lambda_1^s+\max\{\mu(t_1-2(1-\lambda_2)\sin 18^\circ),\mu(t_1-2(1-\lambda_5)\sin 18^\circ)\}}{(r_2+t_1-r_2)^s}\\
&\ge\frac{\lambda_1^s+\max\{\mu(t_1-(1-\lambda_2)2\sin 18^\circ),\mu(t_1-(1-\lambda_5)2\sin 18^\circ)\}}{r_2^s+(t_1-r_2)^s}\\
&\ge\min\left\{\frac{\lambda_1^s}{r_2^s},\frac{\tau}{(t_1-r_2)^s}\right\},
\end{aligned}$$

where

$$\tau=\max\{\mu(t_1-(1-\lambda_2)2\sin 18^\circ),\mu(t_1-(1-\lambda_5)2\sin 18^\circ)\}.$$

This contradicts the assumption that $d(\mu,t_1)$ attains its infimum $d^{(1)}_{\min}$ at the interval $(r_2,r_3]$.

Case 2. $d(\mu,t_1)$ attains its infimum $d^{(1)}_{\min}$ at $(r_4,2\sin 54^\circ]$. In this case, the line E_1F_1 intersects $f_3(E)$ or $f_4(E)$, then

$$\begin{aligned}
d(\mu,t_1) &= \frac{\mu(t_1)}{t_1^s}\\
&= \frac{\lambda_1^s+\lambda_2^s+\lambda_3^s+\max\{\mu(t_1-2\lambda_3\sin 54^\circ),\mu(t_1-2\lambda_4\sin 54^\circ)}{(r_4+t_1-r_4)^s}\\
&\ge \frac{\lambda_1^s+\lambda_2^s+\lambda_3^s+\max\{\mu(t_1-2\lambda_3\sin 18^\circ),\mu(t_1-2\lambda_4\sin 18^\circ)}{(r_4+t_1-r_4)^s}\\
&\ge \frac{\lambda_1^s+\lambda_2^s+\lambda_3^s+\max\{\mu(t_1-2\lambda_3\sin 18^\circ),\mu(t_1-2\lambda_4\sin 18^\circ)}{r_4^s+(t_1-r_4)^s}\\
&\ge \min\left\{\frac{\lambda_1^s}{r_4^s},\frac{\lambda_2^s}{r_4^s},\frac{\lambda_3^s}{r_4^s},\frac{\tau}{(t_1-r_4)^s}\right\},
\end{aligned}$$

where $\tau=\max\{\mu(t_1-2\lambda_3\sin 54^\circ),\mu(t_1-2\lambda_4\sin 18^\circ)\}$.

Case 3. $d(\mu,t_1)$ attains its infimum $d^{(1)}_{\min}$ at $(2\lambda_1\sin 54^\circ,r_2]$. In this case, we have

$$d(\mu,t_1)=\frac{\lambda_1^s}{t_1^s}\ge\frac{\lambda_1^s}{r_2^s}.$$

This means that $t_1=2(1-\lambda_2)\sin 18^\circ$ or $t_1=2(1-\lambda_5)\sin 18^\circ$, and $K=0$.

Case 4. $d(\mu,t_1)$ attains its infimum $d^{(1)}_{\min}$ at the interval$(r_3,r_4]$. We have

$$d(\mu,t_1)=\frac{\lambda_1^s+\lambda_2^s+\lambda_3^s}{t_1^s}\ge\frac{\lambda_1^s+\lambda_2^s+\lambda_3^s}{r_4^s}.$$

Therefore,

$$d(\mu,t_1)=\frac{\lambda_1^s+\lambda_2^s+\lambda_3^s}{r_4^s}.$$

This means that $t_1=\dfrac{1-\lambda_3}{2\sin 18^\circ}$ or $t_1=\dfrac{1-\lambda_4}{2\sin 18^\circ}$, and $K=0$.

Similarly, if $K>0$ and $2\lambda_1^{k+1}\sin 54^\circ<t_1<\lambda_1^k\sin 54^\circ$, we can prove that $d(\mu,t_1)$ attains its infimum $d^{(1)}_{\min}$ only at $t_1=\lambda_1^k\dfrac{1-\lambda_3}{2\sin 18^\circ}$, $\lambda_1^k\dfrac{1-\lambda_4}{2\sin 18^\circ}$, $2\lambda_1^k(1-\lambda_2)\sin 18^\circ$ and $2\lambda_1^k(1-\lambda_5)\sin 18^\circ$.

Lemma 5[3]. *Let* $0<\alpha<1$, $p\le p_0$, $a\ge a_0$, $y\ge\lambda x^\alpha$. *If*

$$0<x\le\left(\frac{a_0\lambda}{p_0}\right)^{\frac{1}{1-\alpha}},$$

then $\dfrac{p-y}{(a-x)^\alpha}<\dfrac{p}{a^\alpha}$.

3 The Main Result

Theorem 6. *Let E be a self-similar set defined by Definition 3, $0 < \lambda_i < \frac{7-2\sqrt{5}}{9}(i = 1,2,3,4,5)$, $s = \dim_H(E)$ and $0 < s < 1$. Moreover, assume the following two conditions*

$$(1) \quad \frac{\lambda_i^s + \lambda_j^s}{(1-\lambda_i-\lambda_j)^s} \le \left(\frac{1}{2\sin 54^\circ}\right)^s, \textit{ for } (i,j) \in M_2;$$

$$(2) \quad 2(\lambda_i+\lambda_j)\sin 54^\circ \le \min\{2d_{\min}^{(i)}\sin 54^\circ, 2d_{\min}^{(j)}\sin 54^\circ\}^{\frac{1}{1-s}}, \textit{ for } (i,j) \in M_1$$

are satisfied. Then for any $x \in E$, if the closed convex set U_x containing x is the closure $\bar{E}_0$ of E_0, then

$$\bar{D}_C^s(E,x) = \sup_{0<|U_x|}\left\{\frac{H^s(E\cap U_x)}{|U_x|^s}\right\} = \frac{H^s(E\cap \bar{E}_0)}{(2\sin 54^\circ)^s} = 1.$$

Proof. Let $V \subset R^2$, $V \bigcap E \ne \varnothing$ and $V \subset \bar{E}_0$(if not, replacing V by $V \bigcap \bar{E}_0$). Denote

$$d(V) = \frac{\mu(V)}{|V|^s}, d_{\max} = \sup_{0<|v|}\{d(V), V \subset \bar{E}_0\},$$

where μ is the mass distribution of E defined as above. We now prove that if $V = \bar{E}_0$ then

$$d_{\max} = \frac{\mu(V)}{|V|^s} = \frac{\mu(\bar{E}_0)}{|\bar{E}_0|^s}.$$

Case 1. $V \bigcap f_i(E) \ne \varnothing$ for all i (See Fig. 2). In this case, we can select fine tangent

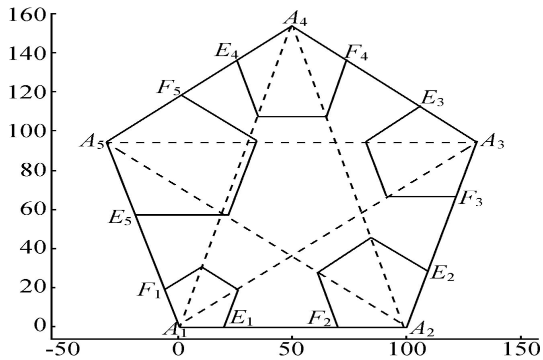

Fig. 2

lines of V, denoted by E_iF_i, such that E_iF_i is parallel to the opposite side of the vertex A_i for $i = 1,2,3,4,5$. Moreover, denote

$$t_i = \frac{d_i}{\cos 18^\circ},$$

where d_i is the distance between the vertex A_i and E_iF_i, then

$$|V| \geq 2\sin 54^\circ - t_i - t_j, \text{ for } (i,j) \in M_1;$$
$$\mu(V) \leq \sum_{i=1}^{5} (\lambda_i^s - \mu(t_i)) = 1 - \sum_{i=1}^{5} \mu(t_i)$$

Therefore

$$\frac{\mu(V)}{|V|^s} \leq \frac{1 - \sum\limits_{i=1}^{5} \mu(t_i)}{(2\sin 54^\circ - t_i - t_j)^s} \leq \frac{1 - (\mu(t_i) + \mu(t_j))}{(2\sin 54^\circ - t_i - t_j)^s}.$$

Replacing α by s, a and a_0 by $2\sin 54^\circ$, p and p_0 by 1, respectively, in Lemma 5, employing Lemma 4, we have

$$\begin{aligned}\frac{\mu(t_i)+\mu(t_j)}{(t_i+t_j)^s} &\geq \frac{\mu(t_i)+\mu(t_j)}{t_i^s+t_j^s} \geq \min\left\{\frac{\mu(t_i)}{t_i^s}, \frac{\mu(t_j)}{t_j^s}\right\} \\ &\geq \min\left\{d_{\min}^{(i)}, d_{\min}^{(j)}\right\}. \triangleq \lambda\end{aligned}$$

Notice the condition (2), we have

$$\begin{aligned}0 < \lambda_i + \lambda_j &\leq \frac{\min\left\{2d_{\min}^{(i)}\sin 54^\circ, 2d_{\min}^{(j)}\sin 54^\circ\right\}^{\frac{1}{1-s}}}{2\sin 54^\circ} \\ &= [(2\sin 54^\circ)^s]^{\frac{1}{1-s}} \min\left\{d_{\min}^{(i)}, d_{\min}^{(j)}\right\}^{\frac{1}{1-s}} \\ &\leq [(2\sin 54^\circ)]^{\frac{1}{1-s}} \min\left\{d_{\min}^{(i)}, d_{\min}^{(j)}\right\}^{\frac{1}{1-s}} = \left(\frac{a_0\lambda}{p_0}\right)^{\frac{1}{1-s}}.\end{aligned}$$

This means that the conditions of Lemma 5 are satisfied. Denote $w = \lambda_i + \lambda_j$, then

$$y = \mu(t_i) + \mu(t_j) \geq \lambda(t_i^s + t_j^s) \geq \lambda(t_i + t_j)^s = \lambda w.$$

Therefore,

$$\frac{\mu(V)}{|V|^s} \leq \frac{p-y}{(a-w)^\alpha} \leq \frac{1}{(2\sin 54^\circ)^s}.$$

That is

$$d_{\max} = \frac{\mu(V)}{|V|^s} = \frac{\mu(\bar{E}_0)}{|\bar{E}_0|^s}.$$

Case 2. There exist only four of five sets $f_i(E)$ such that $V \bigcap f_i(E) \neq \varnothing$. For convenience, let $f_1(E)$, $f_2(E)$, $f_3(E)$ and $f_4(E)$ be these four sets. Then

$$|V| \geq 2\sin 54^\circ - t_1 - t_3,$$
$$|V| \geq 2\sin 54^\circ - t_1 - t_4,$$
$$|V| \geq 2\sin 54^\circ - t_2 - t_4,;$$

and

$$\mu(V) \le 1-\mu(t_1)-\mu(t_3),$$
$$\mu(V) \le 1-\mu(t_1)-\mu(t_4),$$
$$\mu(V) \le 1-\mu(t_2)-\mu(t_4).$$

So,

$$\begin{array}{c} |V| \ge 2\sin 54^\circ - t_i - t_j \\ \mu(V) \le 1-\mu(t_i)-\mu(t_j) \end{array}, \text{ for } (i,j) \in M_3,$$

where $M_3 \triangleq M_1 \backslash \{(2,5),(5,2),(3,5),(5,3)\}$.

Therefore,

$$\frac{\mu(V)}{|V|^s} \le \frac{1-(\mu(t_i)+\mu(t_j))}{(2\sin 54^\circ - t_i - t_j)^s}, (i,j) \in M_3.$$

Employing Lemma 5, we get

$$\frac{\mu(V)}{|V|^s} \le \frac{1}{(2\sin 54^\circ)^s}.$$

That means that the result is still true.

Case 3. There exist only three of five sets $f_i(E)$ such that $V \bigcap f_i(E) \neq \varnothing$. In this case, there exists $(i_0, j_0) \in M_1$ such that

$$V \bigcap f_{i_0}(E) \neq \varnothing \text{ and } V \bigcap f_{j_0}(E) \neq \varnothing.$$

Similar to the proof of Case 2, we deduce that

$$\frac{\mu(V)}{|V|^s} \le \frac{1-(\mu(t_{i_0})+\mu(t_{j_0}))}{(2\sin 54^\circ - t_{i_0} - t_{j_0})^s}.$$

This combining with Lemma follows that

$$\frac{\mu(V)}{|V|^s} \le \frac{1}{(2\sin 54^\circ)^s}.$$

Case 4. There exist only two of five sets $f_i(E)$ such that $V \bigcap f_i(E) \neq \varnothing$. Therefore, we can assume that there exists (i,j) such that

$$V \bigcap f_i(E) \neq \varnothing \text{ and } V \bigcap f_j(E) \neq \varnothing.$$

If $(i,j) \in M_1$, then we get the required result by Case 3. If $(i,j) \in M_2$, and assume $(i,j) = (1,2)$, then

$$\mu(V) \le \lambda_1^s + \lambda_2^s - \mu(t_1) - \mu(t_2) \le \lambda_1^s + \lambda_2^s$$

and

$$|V| \geq 1 - \frac{d_1}{\cos 54^\circ} - \frac{d_2}{\cos 54^\circ} \geq 1 - \lambda_1 - \lambda_2.$$

From the condition (1), we have

$$\frac{\mu(V)}{|V|^s} \leq \frac{\lambda_1^s + \lambda_2^s}{(1-\lambda_1-\lambda_2)^s} \leq \frac{1}{(2\sin 54^\circ)^s}.$$

Case 5. There exists only one of five sets $f_i(E)$ such that $V \bigcap f_i(E) \neq \varnothing$, for example, $V \bigcap f_1(E) \neq \varnothing$. Notice the function of amplification of f_1^{-1}, and

$$\frac{\mu(V \bigcap f_1(E))}{|V \bigcap f_1(E)|^s} = \frac{\mu(f_1^{-1}((V \bigcap f_1(E)))}{\lambda_1^{-1}|V \bigcap f_1(E)|^s}.$$

Denote $V' = f_1^{-1}(V \bigcap f_1(E))$, we can assume

$$V' \bigcap f_i(E) \neq \varnothing,\ V' \bigcap f_j(E) \neq \varnothing$$

for some (i, j) and the density is invariant, if not, then take $f_1^{-1}(V')$ as V'. Similar to the proof of above case, we get the required result.

Therefore,

$$d_{\max} = \frac{\mu(V)}{|V|^s} = \frac{\mu(\bar{E}_0)}{|\bar{E}_0|^s},$$

we finish the proof.

By the definition of probability measure, we know that there exists a constant C such that $\mu = CH^s$. So

$$\bar{D}_C^s(E,x) = \sup_{0<|u_x|} \left\{ \frac{H^s(E \cap U_x)}{|U_x|^s} \right\}$$

attains the supremum at the set $\bar{E}_0$.

Combining Theorem 2.3 in reference [3] and Proposition 2 in reference [4], we get

$$\bar{D}_C^s(E,x) = \sup_{0<|u_x|} \left\{ \frac{H^s(E \cap U_x)}{|U_x|^s} \right\} = \frac{H^s(E \cap \bar{E}_0)}{(2\sin 54^\circ)^s} = 1.$$

Employing Theorem 6, we have the following corollary.

Corollary 7. *If the assumptions in Theorem* 6 *are satisfied, then* $\bar{E}_0$ *is the "best covering" of* E. *That is*

$$H^s(E) = |\bar{E}_0|^s = (2\sin 54^\circ)^s,$$

where $s = \dim_H(E)$ *satisfies*

$$\sum_{i=1}^{5} \lambda_i^s = 1.$$

4 Examples

Example 8. Let $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \dfrac{1}{25}$, then $s = \dfrac{1}{2}$. Moreover, we have
(1) $\dfrac{\lambda_i^s + \lambda_j^s}{(1-\lambda_i-\lambda_j)^s} = \dfrac{2}{23} \approx 0.0870 \le \left(\dfrac{1}{2\sin 54^\circ}\right)^s = 0.7862$, for $(i,j) \in M_2$;
(2) for$(i,j) \in M_1, 2\sin 54^\circ(\lambda_i+\lambda_j) \approx 0.1294$,

$$d_{\min}^{(i)} = \left(\frac{2\sin 54^\circ}{24}\right)^s \approx 0.2596,$$

$$\min\left\{2\sin 54^\circ \cdot d_{\min}^{(i)}, 2\sin 54^\circ \cdot d_{\min}^{(j)}\right\}^{\frac{1}{1-s}}$$
$$= \left(\frac{(2\sin 54^\circ)^{s+1}}{24^s}\right)^{\frac{1}{1-s}} = 0.1765.$$

Hence, the assumptions of Theorem 6 are satisfied. Therefore,

$$H^s(E) = (2\sin 54^\circ)^s \approx 1.272.$$

Example 9. Let $\lambda_1 = \lambda_3 = \lambda_5 = \dfrac{1}{625}$, $\lambda_2 = \lambda_4 = \dfrac{1}{25}$. Since $\sum_{i=1}^{5} \lambda_i = 1$, then $2\left(\frac{1}{25}\right)^s + 3\left(\frac{1}{625}\right)^s = 1$. Denote $x = \left(\dfrac{1}{25}\right)^s$, then $3x^2 + 2x - 1 =$ and $s = \dfrac{1}{2}\log_5 3$. Then,

(1) $\left(\dfrac{1}{2\sin 54^\circ}\right)^s = \left(\dfrac{1}{2\sin 54^\circ}\right)^{\frac{1}{2}\log_5 3} = 0.7862,$ and for $(i,j) \in M_2$,

$$\frac{\lambda_i^s + \lambda_j^s}{(1-\lambda_i-\lambda_j)^s} = \frac{\left(\frac{1}{25}\right)^s + \left(\frac{1}{25}\right)^s}{(1-\frac{1}{25}-\frac{1}{25})^s} \approx 0.6859,$$

$$\text{or} \quad \frac{\lambda_i^s + \lambda_j^s}{(1-\lambda_i-\lambda_j)^s} = \frac{\left(\frac{1}{25}\right)^s + \left(\frac{1}{625}\right)^s}{(1-\frac{1}{25}-\frac{1}{625})^s} \approx 0.4768,$$

$$\text{or} \quad \frac{\lambda_i^s + \lambda_j^s}{(1-\lambda_i-\lambda_j)^s} = \frac{\left(\frac{1}{625}\right)^s + \left(\frac{1}{625}\right)^s}{(1-\frac{1}{625}-\frac{1}{625})^s} \approx 0.2225.$$

(2) for $(i,j) \in M_1, 2\sin 54^\circ(\lambda_i+\lambda_j) \approx 0.0337$ or 0.0052,

$$d_{\min}^{(1)} = d_{\min}^{(3)} = d_{\min}^{(5)} = \left(\frac{2\sin 54^\circ}{624}\right)^{\frac{1}{2}\log_5^3} \approx 0.1309,$$

$$d_{\min}^{(2)} = d_{\min}^{(4)} = \left(\frac{2\sin 54^\circ}{24}\right)^{\frac{1}{2}\log_5^3} \approx 0.3982,$$

$$\min\left\{2d_{\min}^{(i)}\sin 54^\circ, 2d_{\min}^{(j)}\sin 54^\circ\right\}^{\frac{1}{1-s}}$$
$$= ((2\sin 54^\circ)\cdot 0.1309)^{\frac{1}{1-s}} = 0.0944.$$

So the assumptions of Theorem 6 are satisfied. Therefore,

$$H^s(E) = (2\sin 54^\circ)^s = (2\sin 54^\circ)^{\log_5 \sqrt{3}} \approx 1.183.$$

References

[1] Zhou, Z. L. and Feng, L., Twelve Open Problems on the Exact Value of the Hausdorff Measure and on Topological Entropy: a Brief Survey of Recent Results, Nonlinearity, 17(2004), 493-502.

[2] Zhu, Z. W., Zhou, Z. L. and Jia, B. G., The Hausdorff Measure and Upper Convex Density of a Class of Self-similar Sets on the Plane, Acta Mathematica Sinica, Chinese Series, 48:3(2005), 535-540.

[3] Ayer, E. and Strichartz, R. S., Exact Hausdorff Measure and Intervals of Maximum Density for Cantor Sets, Trans. Amer. Math. Soc., 351:9 (1999), 3725-3741

[4] Zhou, Z. L., Hausdorff Measure of Self-similar Sets-koch cure, Science of China (Ser A), 28:2(1998), 103-107.

[5] Falconer, K. J., The Geometry of Fractal sets, Cambridge University press, Cambridge (1985).

Department of Mathematics and Information Technology
Hanshan Normal University
Chaozhou, 521041
P. R. China

C. Y. Zeng

E-mail: zcy@hstc.ecu.cn

D. H. Yuan

E-mail: ydhlxl@hstc.edu.cn

S. Y. Xu
School of Mathematics and Computer
Gannan Normal University
Ganzhou, 341000

E-mail: xushaoyuan@126.com