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On Approximation by Reciprocals of Polynomials with Positive Coefficients

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Abstract. In order to study the approximation by reciprocals of polynomials with real coefficients, one always assumes that the approximated function has a fixed sign on the given interval. Sometimes, the approximated function is permitted to have finite sign changes, such as $l(l\geq 1)$ times. Zhou Songping has studied the case $l=1$ and $l\geq 2$ in L^p spaces in order of priority. In this paper, we studied the case $l\geq 2$ in Orlicz spaces by using the function extend, modified Jackson kernel, Hardy-Littlewood maximal function, Cauchy-Schwarz inequality, and obtained the Jackson type estimation.

Key Words: Approximation, polynomial, Steklov function, Orlicz space, modulus of continuity.
AMS Subject Classifications: 41A17, 41A20

1 Introduction and main result

Denote by $\Pi_n(+)$ the set of all polynomials with positive coefficients of degree n, that is

$$\Pi_n(+)=\left\{P_n(x):P_n(x)=\sum_{0\leq k+l\leq n} a_{k,l}x^k(1-x)^l,a_{k,l}>0\right\}.$$

In order to consider approximation by reciprocals of polynomials with real coefficients,we always assume that the given function f has a fixed sign on the given interval. In general,we allow the function f to have finite sign changes, such as $l(l\geq 1)$ times, and this result was first given by Leviatan, Lubinsky in [1]. They proved the following.

Theorem 1.1. *Let $f(x)\in C_{[-1,1]}$ change its sign exactly l times in $(-1,1)$, say at $-1<b_1<b_2<\cdots<b_l<1$, then for each $n\geq 1$, there exists $P_n(x)\in\Pi_n(+)$ having the same sign as f in $(b_l,1)$, such that for $x\in[-1,1]$*

$$\left\|f(x)-\frac{\prod_{j=1}^{l}(x-b_j)}{p_n(x)}\right\|_C\leq C(l+1)^2\omega\Big(f,\frac{1}{n}\Big)_C.$$

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In [2], Zhou partly generalized the result in [1] for the case $l=1$, $1<p<\infty$. In [3], Wang and Wu generalized the result in [2] to Orlicz spaces. In [4], Zhou and Mei studied the case $f(x)\in L^p_{[0,1]}$ $(1<p<\infty)$ and have sign changes $l(l\geq 2)$ times, they obtained

Theorem 1.2. *Let* $f(x)\in L^p_{[0,1]}$ $(1<p<\infty)$, *and change sign exactly* $l(l\geq 2)$ *times in* $(0,1)$, *then there exist* $0<b_1<b_2<\cdots<b_l<1$ *and* $P_n(x)\in\Pi_n(+)$, *such that*

$$\left\|f(x)-\frac{\prod_{j=1}^{l}(x-b_j)}{p_n(x)}\right\|_{L^p_{[0,1]}}\leq C_{p,b,l}\,\omega(f,n^{-\frac{1}{2}})_{L^p_{[0,1]}},$$

where $b=\min\{|b_{j+1}-b_j|: j=1,2,\cdots,l-1\}$, $C_{p,b,l}$ *is a positive constant depending only on* p, b *and* l.

In this paper we consider the similar problem in Orlicz spaces.

Let $M(u)$ and $N(v)$ be mutually complementary N functions, the definition and properties of N function can be seen in [5]. The Orlicz space $L^*_{M(G)}$ corresponding to the N function $M(u)$ consists of all Lebesgue measurable functions $u(x)$ on G, of which the Orlicz norm

$$\|u\|_M=\sup_{\rho(v,N)\leq 1}\left|\int_G u(x)v(x)\mathrm{d}x\right| \tag{1.1}$$

is finite, here

$$\rho(v,N)=\int_G N(v(x))\mathrm{d}x$$

is the modulus of $v(x)$ with respect to $N(v)$. According to [5], the Orlicz norm (1.1) can also be calculated by

$$\|u\|_M=\inf_{\alpha>0}\frac{1}{\alpha}\left(1+\int_G M(\alpha u(x))\mathrm{d}x\right). \tag{1.2}$$

Define the modulus of smoothness of the function $f(x)\in L^*_{M(G)}$ as

$$\omega(f,t)_M=\sup_{0\leq h\leq t}\|f(\cdot+h)-f(\cdot)\|_{M(I_h)},$$

where $I_h=[0,1-h]$ and $0\leq t<1$.

Definition 1.1. Let $f(x)\in L^*_{M[0,1]}$, we say $f(x)$ changes its sign exactly l times at $a_1,a_2,\cdots,a_l$, if there exist l points $0<a_1<a_2<\cdots<a_l<1$, such that

$$\sigma\,\mathrm{sgn}\Big(\prod_{j=1}^{l}(x-a_j)\Big)f(x)>0 \quad a.e. \quad x\in[0,1], \quad \sigma=\pm 1,$$

and such that for every $j=1,2,\cdots,l$, any $0<\eta<a_{j+1}-a_j$ $(a_{l+1}=1)$,

$$\mathrm{meas}\big(\{x\in(a_j,a_{j+1}): f(x)\neq 0\}\cap(a_j,a_{j+\eta})\big)>0,$$

where we require $\mathrm{meas}\{x\in[0,a_1]: f(x)\neq 0\}>0$.

The main result of this paper is as follows.

Theorem 1.3. *Let $M(u)$ satisfy Δ_2 condition, $f(x)\in L^*_{M[0,1]}$, $f(x)\neq 0$ and change its sign $l(l\geq 2)$ times in $(0,1)$, then there exist $0<b_1<b_2<\cdots<b_l<1$ and $P_n(x)\in\Pi_n(+)$, such that*

$$\left\|f(x)-\frac{\prod_{j=1}^{l}(x-b_j)}{p_n(x)}\right\|_M\leq C_{b,l}\,\omega(f,n^{-\frac{1}{2}})_M,$$

where $b=\min\{|b_{j+1}-b_j|: j=1,2,\cdots,l-1\}$, $C_{b,l}$ is a positive constant depending only on b, l, $\omega(f,n^{-\frac{1}{2}})_M$ is the modulus of smoothness in the space $L^*_{M(G)}$.

Throughout the paper, we always use C to indicate a positive constant, but their values may vary in different occurrences. Write

$$\|f\|_{M[0,1]}=\|f\|_M,\qquad \omega(f,t)_{M[0,1]}=\omega(f,t)_M.$$

2 Lemmas

Lemma 2.1 ([3]). *Let $f(x)\in L^*_{M[0,1]}$, and extend $f(x)$ as follows:*

$$F_m(x)=\begin{cases} m\displaystyle\int_0^{\frac{1}{m}}f(t)\mathrm{d}t, & x\in[-2,0),\\ f(x), & x\in[0,1],\\ m\displaystyle\int_{1-\frac{1}{m}}^{1}f(t)\mathrm{d}t, & x\in(1,3],\end{cases}$$

then we have $F_m(x)\in L^*_{M[-2,3]}$, and

$$\omega(F_m,m^{-1})_{M[-1,2]}\leq C\omega(f,m^{-1})_M.$$

Lemma 2.2 ([3]). *Let $f(x)\in L^*_{M[0,1]}$ change its sign exactly l times in $(0,1)$, and write*

$$f_h(x)=\frac{1}{h}\int_{x-\frac{h}{2}}^{x+\frac{h}{2}}f(u)\mathrm{d}u$$

for the Steklov function of order one of $f(x)$, then for sufficiently small $h>0$, $f_h(x)$ changes its sign l times in $(\frac{h}{2},1-\frac{h}{2})$.

Lemma 2.3 ([3]). *Let $f(x)\in L^*_{M[0,1]}$, $f_h(x)$ be the Steklov function of order one of $f(x)$, define*

$$f_{hh}(x)=\frac{1}{h}\int_{x-\frac{h}{2}}^{x+\frac{h}{2}}f_h(u)\mathrm{d}u$$

as the Steklov function of order two of $f(x)$, then for sufficiently small $h>0$, we have

$$\|f-f_h\|_{M[\frac{h}{2},1-\frac{h}{2}]}\le C\omega(f,h)_M,$$
$$\|f-f_{hh}\|_{M[h,1-h]}\le C\omega(f,h)_M,$$
$$\|f'_{hh}\|_{M[h,1-h]}\le Ch^{-1}\omega(f,h)_M,$$
$$\|f''_{hh}\|_{M[h,1-h]}\le Ch^{-2}\omega(f,h)_M.$$

Lemma 2.4 ([6]). *Let $f(x)\in L^*_{M[a,b]}$, $x\in I\subset[a,b]$, and define*

$$M(f,x)=\sup_{x\in I}\frac{1}{|I|}\int_I|f(u)|\mathrm{d}u,$$

then

$$\|M(f)\|_{M[a,b]}\le C\|f\|_{M[a,b]}.$$

Lemma 2.5 ([3]). *Define*

$$\lambda_n(t)=C_n\left\{\left(\frac{\sin\frac{n(t-\delta_n)}{2}}{\sin\frac{t-\delta_n}{2}}\right)^4+\left(\frac{\sin\frac{n(t+\delta_n)}{2}}{\sin\frac{t+\delta_n}{2}}\right)^4\right\}$$

as the modified Jackson kernel, where $\delta_n=\frac{\pi}{2n}$ and C_n is taken such that $\int_{-\pi}^{\pi}\lambda_n(t)dt=1$. Let $f(x)$ be a power integrable function with the period 2π and define

$$\Lambda_n(f,x)=\int_{-\pi}^{\pi}f(x+s)\lambda_n(s)\mathrm{d}s,$$

then we have

$$\|f-\Lambda_n(f)\|_{M,2\pi}\le C\omega\Big(f,\frac{1}{n}\Big)_{M,2\pi},$$
$$\omega(\Lambda_n(f),t)_{M,2\pi}\le C\omega(f,t)_{M,2\pi},$$
$$\sup_{-\pi\le x\le\pi}\frac{\Lambda_n(f,x)}{\Lambda_n(f,x+t)}\le C(1+n|t|)^4,$$
$$\int_{-\pi}^{\pi}t^j\lambda_n(t)dt\sim n^{-j},\qquad j=0,1,2.$$

Lemma 2.6 ([4]). *Let $l\ge1$, $-1<b_1<b_2<\cdots<b_l<1$, $b=\min\{b_{j+1}-b_j:j=1,2,\cdots,l-1\}$, then for given numbers A and B, $B\neq b_j$, the following inequality holds:*

$$\left|1-\frac{\prod_{j=1}^l(A-b_j)}{\prod_{j=1}^l(B-b_j)}\right|\le C_{b,l}M_l\sum_{j=1}^l\frac{1}{|B-b_j|},$$

where

$$M_l=\begin{cases}|B-A|, & |B-A|\le1,\\ |B-A|^l, & |B-A|>1.\end{cases}$$

3 Proof of Theorem 1.3

Assume $l\geq 2$, $f(x)\geq 0$ and extend $f(x)\in L^*_{M[0,1]}$ to $F_m(x)\in L^*_{M[-2,3]}$ in the way described in Lemma 2.1. Since $f(x)$ changes its sign l times in $(0,1)$, obviously $F_m(x)$ changes its sign l times in $(-2,3)$, and satisfies

$$\omega(F_m,m^{-1})_{M[-1,2]}\leq C\omega(f,m^{-1})_M.$$

Take a sufficiently small $h>0$, for $x\in[-1,2]$, we define the second order Steklov function $\hat{F}_m$ for $F_m(x)$, that is $\hat{F}_m(x)=(F_m(x))_{hh}$ (see the definition in Lemma 2.3). Corresponding to the l sign change points $a_1<a_2<\cdots<a_l$ of $f(x)$ in $(0,1)$, from Lemma 2.2 we have l points $0<b_1<b_2<\cdots<b_l<1$ at which $\hat{F}_m(x)$ changes its sign. Assume

$$\text{sgn}\prod_{j=1}^{l}(x-b_j)\hat{F}_m(x)\geq 0,$$

for given $\epsilon>0$, $x\in[0,1]$ ($[0,1]\subset[-1+h,2-h]$), set

$$g(x)=\frac{\hat{F}_m(x)}{\prod_{j=1}^{l}(x-b_j)}+\epsilon,\qquad G(\theta)=g\Big(\frac{3\cos\theta+1}{2}\Big),$$
$$\widetilde{g}(x)=\Lambda_m(G,\theta)=\int_{-\pi}^{\pi}g\Big(\frac{3\cos(\theta+s)+1}{2}\Big)\lambda_m(s)ds.$$

Write $I_k=[\frac{k}{n+1},\frac{k+1}{n+1}]$, $k=0,1,2,\cdots,n$. As usual, for $h(x)\in L^*_{M[0,1]}$,

$$B_n(h,x)=\sum_{k=0}^{n}(n+1)\int_{I_k}h(u)\mathrm{d}u\,p_{n,k}(x)$$

is the Kantorovich polynomial of degree n.

Define

$$P_n(x)=B_n\Big(\frac{1}{\tilde{g}},x\Big)=\sum_{k=0}^{n}(n+1)\int_{I_k}\frac{1}{\tilde{g}(t)}\mathrm{d}t\,p_{n,k}(x),$$

then $P_n(x)\in\Pi_n(+)$. We take $m=[n^{\frac{1}{2}}]$, $\epsilon=\omega(f,n^{-\frac{1}{2}})_M$, $h=n^{-\frac{1}{2}}$,

$$\begin{aligned}&\Big\|f(x)-\frac{\prod_{j=1}^{l}(x-b_j)}{p_n(x)}\Big\|_M\\&\leq\Big\|f(x)-\Big(\prod_{j=1}^{l}(x-b_j)\Big)g(x)\Big\|_M\\&\quad+\Big\|\Big(\prod_{j=1}^{l}(x-b_j)\Big)(g(x)-\tilde{g}(x))\Big\|_M+\Big\|\Big(\prod_{j=1}^{l}(x-b_j)\Big)\Big(\tilde{g}(x)-\frac{1}{p_n(x)}\Big)\Big\|_M\\&=\|I_1\|_M+\|I_2\|_M+\|I_3\|_M.\end{aligned}$$

By Lemma 2.1, Lemma 2.3 and $\epsilon=\omega(f,n^{-\frac{1}{2}})_M$, we obtain

$$\begin{aligned}\|I_1\|_M&=\Big\|f(x)-\hat{F_m}(x)-\epsilon\prod_{j=1}^{l}(x-b_j)\Big\|_M\\&\le\Big\|F_m(x)-\hat{F_m}(x)-\epsilon\prod_{j=1}^{l}(x-b_j)\Big\|_{M[-1+h,2-h]}\\&\le C(\omega(F_m,h)_{M[-1,2]}+\epsilon)\le C\omega(f,n^{-\frac{1}{2}})_M.\end{aligned}\tag{3.1}$$

At the same time, for $\|I_2\|_M$, we have the estimation

$$\begin{aligned}\|I_2\|_M=&\Big\|\Big(\prod_{j=1}^{l}(x-b_j)\Big)(g(x)-\tilde{g}(x))\Big\|_M\\=&\Big\|\int_{-\pi}^{\pi}\Big(\hat{F_m}\Big(\frac{3\cos\theta+1}{2}\Big)-\frac{\prod_{j=1}^{l}\big(\frac{3\cos\theta+1}{2}-b_j\big)}{\prod_{j=1}^{l}\big(\frac{3\cos(\theta+s)+1}{2}-b_j\big)}\hat{F_m}\Big(\frac{3\cos(\theta+s)+1}{2}\Big)\Big)\lambda_m(s)ds\Big\|_M\\\le&\Big\|\int_{-\pi}^{\pi}\Big(\hat{F_m}\Big(\frac{3\cos\theta+1}{2}\Big)-\hat{F_m}\Big(\frac{3\cos(\theta+s)+1}{2}\Big)\Big)\lambda_m(s)ds\Big\|_M\\&+\Big\|\int_{-\pi}^{\pi}\Big(1-\frac{\prod_{j=1}^{l}\big(\frac{3\cos\theta+1}{2}-b_j\big)}{\prod_{j=1}^{l}\big(\frac{3\cos(\theta+s)+1}{2}-b_j\big)}\Big)\hat{F_m}\Big(\frac{3\cos(\theta+s)+1}{2}\Big)\lambda_m(s)ds\Big\|_M\\=&\|I_{21}\|_M+\|I_{22}\|_M.\end{aligned}$$

By [4], we know

$$\begin{aligned}&|I_{21}|\le Cn^{-\frac{1}{2}}M(\hat{F'_m},x),\\&|I_{22}|\le C_{b,l}\big(n^{-\frac{1}{2}}M(\hat{F'_m},x)+n^{-\frac{1}{2}}|\hat{F'_m}(x)|+n^{-1}M(\hat{F''_m},x)\big).\end{aligned}$$

By Lemmas 2.1, 2.3, 2.4 and $h=n^{-\frac{1}{2}}$, $m=[n^{\frac{1}{2}}]$ we obtain

$$\begin{aligned}\|I_2\|_M\le&\|I_2\|_{M[-1+h,2-h]}\le\|I_{21}\|_{M[-1+h,2-h]}+\|I_{22}\|_{M[-1+h,2-h]}\\\le&C_{b,l}\Big(n^{-\frac{1}{2}}\big(\|M(\hat{F'_m},x)\|_{M[-1+h,2-h]}\\&\qquad+\|\hat{F'_m}(x)\|_{M[-1+h,2-h]}\big)+n^{-1}\|M(\hat{F''_m},x)\|_{M[-1+h,2-h]}\Big)\\\le&C_{b,l}\big(n^{-\frac{1}{2}}\|\hat{F'_m}(x)\|_{M[-1+h,2-h]}+n^{-1}\|\hat{F''_m}(x)\|_{M[-1+h,2-h]}\big)\\\le&C_{b,l}\big(n^{-\frac{1}{2}}h^{-1}\omega(F_m,h)_{M[-1,2]}+n^{-1}h^{-2}\omega(F_m,h)_{M[-1,2]}\big)\\\le&C_{b,l}\omega(f,n^{-\frac{1}{2}})_M.\end{aligned}\tag{3.2}$$

To estimate $\|I_3\|_M$ we consider two sets,

$$E_1=\Big\{x\in[0,1]:\frac{1}{p_n(x)}\ge\tilde{g}(x)\Big\},\qquad E_2=[0,1]\backslash E_1.$$

For $x\in E_1$, from Cauchy-Schwarz inequality we obtain

$$B_n(\tilde{g},x)B_n\Big(\frac{1}{\tilde{g}},x\Big)\geq B_n^2(1,x)=1,$$
$$B_n(\tilde{g},x)\geq\frac{1}{B_n(\frac{1}{\tilde{g}},x)}=\frac{1}{p_n(x)}.$$

Hence, for $x\in E_1$, set $t=\frac{3\cos\eta+1}{2}$, then we have

$$\begin{aligned}
|I_3|=&\Big|\prod_{j=1}^{l}(x-b_j)\Big(\tilde{g}(x)-\frac{1}{p_n(x)}\Big)\Big|\leq\Big|\prod_{j=1}^{l}(x-b_j)(\tilde{g}(x)-B_n(\tilde{g},x))\Big|\\
\leq&\sum_{k=0}^{n}(n+1)\int_{I_k}\Big|\prod_{j=1}^{l}(x-b_j)(\Lambda_m(G,\theta)-\Lambda_m(G,\eta))\Big|\mathrm{d}tp_{n,k}(x)\\
\leq&\Big|\sum_{k=0}^{n}(n+1)\int_{I_k}\int_{-\pi}^{\pi}\Big(\hat{F_m}\Big(\frac{3\cos(\theta+s)+1}{2}\Big)-\hat{F_m}\Big(\frac{3\cos(\eta+s)+1}{2}\Big)\Big)\lambda_m(s)\mathrm{d}s\mathrm{d}tp_{n,k}(x)\Big|\\
&+\sum_{k=0}^{n}(n+1)\int_{I_k}\Big|\int_{-\pi}^{\pi}\Big(1-\frac{\prod_{j=1}^{l}(\frac{3\cos\theta+1}{2}-b_j)}{\prod_{j=1}^{l}(\frac{3\cos(\theta+s)+1}{2}-b_j)}\Big)\hat{F_m}\Big(\frac{3\cos(\theta+s)+1}{2}\Big)\lambda_m(s)\mathrm{d}s\Big|\mathrm{d}tp_{n,k}(x)\\
&+\sum_{k=0}^{n}(n+1)\int_{I_k}\Big|\int_{-\pi}^{\pi}\Big(1-\frac{\prod_{j=1}^{l}(\frac{3\cos\theta+1}{2}-b_j)}{\prod_{j=1}^{l}(\frac{3\cos(\eta+s)+1}{2}-b_j)}\Big)\hat{F_m}\Big(\frac{3\cos(\eta+s)+1}{2}\Big)\lambda_m(s)\mathrm{d}s\Big|\mathrm{d}tp_{n,k}(x)\\
=&|I_{31}|+|I_{32}|+|I_{33}|.
\end{aligned}$$

By [4] we know

$$\begin{aligned}
&|I_{31}|\leq Cn^{-\frac{1}{2}}M(\hat{F}'_m,x),\\
&|I_{32}|\leq C_{b,l}\big(n^{-\frac{1}{2}}|\hat{F}'_m(x)|+n^{-\frac{1}{2}}M(\hat{F}'_m,x)+n^{-1}M(\hat{F}''_m,x)\big),\\
&|I_{33}|\leq C_{b,l}\big(n^{-\frac{1}{2}}|\hat{F}'_m(x)|+n^{-\frac{1}{2}}M(\hat{F}'_m,x)+n^{-1}M(\hat{F}''_m,x)\big).
\end{aligned}$$

Hence, for $x\in E_1$,

$$\begin{aligned}
&\|I_3\|_{M(E_1)}\leq\|I_3\|_M\leq\|I_3\|_{M[-1+h,2-h]}\\
\leq&\|I_{31}\|_{M[-1+h,2-h]}+\|I_{32}\|_{M[-1+h,2-h]}+\|I_{33}\|_{M[-1+h,2-h]}\\
\leq&C_{b,l}\big(n^{-\frac{1}{2}}(\|\hat{F}'_m(x)\|_{M[-1+h,2-h]}+\|M(\hat{F}'_m,x)\|_{M[-1+h,2-h]})+n^{-1}\|M(\hat{F}''_m,x)\|_{M[-1+h,2-h]}\big)\\
\leq&C_{b,l}\omega(F_m,n^{-\frac{1}{2}})_{M[-1,2]}\leq C_{b,l}\omega(f,n^{-\frac{1}{2}})_M.
\end{aligned}$$

On the other hand, for $x\in E_2$ from $\frac{1}{p_n(x)}<\tilde{g}(x)$ we have

$$\begin{aligned}
0\leq\tilde{g}(x)-\frac{1}{p_n(x)}=&\sum_{k=0}^{n}(n+1)\int_{I_k}\frac{[\tilde{g}(x)-\tilde{g}(t)]}{p_n(x)\tilde{g}(t)}\mathrm{d}tp_{n,k}(x)\Big|\tilde{g}(x)-\frac{1}{p_n(x)}\Big|\\
\leq&\sum_{k=0}^{n}(n+1)\int_{I_k}|\tilde{g}(t)-\tilde{g}(x)|\frac{\tilde{g}(x)}{\tilde{g}(t)}\mathrm{d}tp_{n,k}(x).
\end{aligned}$$

For $t\in I_k$, we have

$$\begin{aligned}\frac{\tilde{g}(x)}{\tilde{g}(t)}&=\frac{\Lambda_m(G,\theta)}{\Lambda_m(G,\eta)}\le C(1+m|\theta-\eta|)^4\\&\le C\Big(1+n^2\Big|\arccos\frac{2x-1}{3}-\arccos\frac{2t-1}{3}\Big|^4\Big)\\&\le C(1+n^2|x-t|^4)\le C\Big(1+n^2|x-\frac{k}{n}|^4\Big).\end{aligned}$$

For $x\in E_2$, we have

$$\begin{aligned}|I_3|=&\Big|\Big(\prod_{j=1}^{l}(x-b_j)\Big)(\tilde{g}(x)-\frac{1}{p_n(x)})\Big|\\\le& C\Big|\sum_{k=0}^{n}(n+1)\int_{I_k}\int_{-\pi}^{\pi}\Big(\hat{F_m}\Big(\frac{3\cos(\theta+s)+1}{2}\Big)-\hat{F_m}\Big(\frac{3\cos(\eta+s)+1}{2}\Big)\Big)\lambda_m(s)dsdt\Big|\\&\quad\times\Big(1+n^2\Big|x-\frac{k}{n}\Big|^4\Big)p_{n,k}(x)\\&+C_{b,l}\sum_{k=0}^{n}(n+1)\int_{I_k}\sum_{j=1}^{l}\int_{-\pi}^{\pi}\left|\frac{\hat{F_m}(\frac{3\cos(\theta+s)+1}{2})}{\frac{3\cos(\theta+s)+1}{2}-b_j}\right||s|\lambda_m(s)dsdt\times\Big(1+n^2\Big|x-\frac{k}{n}\Big|^4\Big)p_{n,k}(x)\\&+C_{b,l}\sum_{k=0}^{n}(n+1)\int_{I_k}\sum_{j=1}^{l}\int_{-\pi}^{\pi}\left|\frac{\hat{F_m}(\frac{3\cos(\eta+s)+1}{2})}{\frac{3\cos(\eta+s)+1}{2}-b_j}\right|(\cos\theta-\cos(\eta+s))\lambda_m(s)dsdt\\&\quad\times\Big(1+n^2\Big|x-\frac{k}{n}\Big|^4\Big)p_{n,k}(x)\\=&|\tilde{I_{31}}|+|\tilde{I_{32}}|+|\tilde{I_{33}}|.\end{aligned}$$

By [4] we know

$$\begin{aligned}&|\tilde{I_{31}}|\le Cn^{-\frac{1}{2}}M(\hat{F'_m},x),\\&|\tilde{I_{32}}|\le C_{b,l}\big(n^{-\frac{1}{2}}|\hat{F'_m}(x)|+n^{-\frac{1}{2}}M(\hat{F'_m},x)+n^{-1}M(\hat{F''_m},x)\big),\\&|\tilde{I_{33}}|\le C_{b,l}\big(n^{-\frac{1}{2}}|\hat{F'_m}(x)|+n^{-\frac{1}{2}}M(\hat{F'_m},x)+n^{-1}M(\hat{F''_m},x)\big).\end{aligned}$$

Hence, for $x\in E_2$, we have

$$\begin{aligned}\|I_3\|_{M(E_2)}&\le\|I_3\|_M\le\|I_3\|_{M[-1+h,2-h]}\\&\le\|\tilde{I}_{31}\|_{M[-1+h,2-h]}+\|\tilde{I}_{32}\|_{M[-1+h,2-h]}+\|\tilde{I}_{33}\|_{M[-1+h,2-h]}\\&\le C_{b,l}\Big(n^{-\frac{1}{2}}(\|\hat{F'_m}(x)\|_{M[-1+h,2-h]}\\&\qquad+\|M(\hat{F'_m},x)\|_{M[-1+h,2-h]})+n^{-1}\|M(\hat{F''_m},x)\|_{M[-1+h,2-h]}\Big)\\&\le C_{b,l}\omega(F_m,n^{-\frac{1}{2}})_{M[-1,2]}\le C_{b,l}\omega(f,n^{-\frac{1}{2}})_M.\end{aligned}$$

So, for $x\in[0,1]$, we have

$$\|I_3\|_M\le C_{b,l}\omega(f,n^{-\frac{1}{2}})_M. \tag{3.3}$$

Finally, from (3.1), (3.2) and (3.3), we get

$$\left\|f(x)-\frac{\prod_{j=1}^{l}(x-b_j)}{p_n(x)}\right\|_M\le C_{b,l}\omega(f,n^{-\frac{1}{2}})_M.$$

The theorem is proved. □

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References

[1] Leviatan, D. and Lubinsky, D. S., Degree of approximation by rational function with prescribed numerator degree, Canad. J. Math., 46(1994), 619-633.
[2] Zhou, S. P., Approximation by reciprocals of polynomials with positive coefficients, Southeast Asian Bull. Math., 28:4(2004), 773-781.
[3] Wang, X. L. and Wu, G. R. D., On approximation by reciprocals of polynomials with positive coefficients in Orlicz spaces, Analysis in Theory and Applications, 24:4(2008), 364-377.
[4] Zhou, S. P. and Mei, X. F., The Jackson estimation of approximation by reciprocals of polynomials with positive coefficients in $L^p_{[0,1]}$ spaces for $1<p<\infty$, Advances In Mathematics, 38:2(2009), 241-252.
[5] Wu, C. X. and Wang, T. F., Orlicz Spaces and its Application, HaErBin: HeiLongJiang Science and Technology Press, 1983.
[6] Xie, D. L., Degree of approximation of continuity positive operators in L^*_M, Journal of HangZhou University, 8:2(1981), 142-146.
[7] Wu, X. H. and Wu, G. R. D., An approximation by reciprocals of polynomials with positive coefficients in Orlicz spaces, Journal of Inner Mongolia University: Natural Science Edition, 43:4(2012), 337-341.