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Fixed Point of Multivalued Operators on Partial Metric Spaces

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Abstract. In this paper, we prove some results on fixed point of multivalued operators on partial metric spaces.

Key Words: Fixed point, multivalued operator, partial metric space.

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1 Introduction

In 1992, Matthews introduced partial metric spaces as a generalization of the metric space [8]. In the partial metric space the distance of a point from itself is not necessarily zero [8]. Recently, these spaces have been considered by some authors [1,2,11]. There are a lot of fixed and common fixed point results in different types of spaces. After the remarkable contribution of Matthews, many authors have studied on partial metric spaces and its topological properties (see, e.g., [4–7,10,12,13]. Then, Valero [14], Oltra and Valero [9] and Altun et al. [1] gave some generalizations of the result of Matthews. Also, Romaguera proved a Kirk type fixed point theorem on partial metric spaces [11].

A partial metric is a function $p:X\times X\to[0,\infty)$ satisfying the following conditions:

(a) $p(x,y)=p(y,x)$,

(b) If $p(x,x)=p(x,y)=p(y,y)$, then $x=y$,

(c) $p(x,x)\le p(x,y)$,

(d) $p(x,z)+p(y,y)\le p(x,y)+p(y,z)$,

for all $x,y,z\in X$. A space X with a partial metric p is called a partial metric space denoted by (X,p). If p is a partial metric p on X, then the function $d_p:X\times X\to[0,\infty)$ given by

$$d_p(x,y)=2p(x,y)-p(x,x)-p(y,y)$$

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is a metric on X. Also, each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x,\varepsilon): x\in X,\ \varepsilon>0\}$, where $B_p(x,\varepsilon)=\{y\in X: p(x,y)< p(x,x)+\varepsilon\}$ (see, e.g., [1,2,11]).

Definition 1.1 ([2,8]). Let (X,p) be a partial metric space.

(i) A sequence $\{x_n\}$ converges to $x\in X$ whenever $\lim_{n\to\infty} p(x_n,x)=p(x,x)$;

(ii) $\{x_n\}$ is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and finite);

(iii) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x\in X$, that is, $\lim_{n,m\to\infty} p(x_n,x_m)=p(x,x)$.

Definition 1.2. Let (X,p) be a partial metric space and $T:X\to X$ a selfmap. We say that T is orbitally continuous whenever $\lim_{n\to\infty} p(x_n,x)=p(x,x)$ implies that $\lim_{n\to\infty} p(Tx_n,Tx)=p(Tx,Tx)$.

We need the following results.

Lemma 1.1 ([2,8]). *Let (X,p) be a partial metric space. Then*

(a) *$\{x_n\}$ is a Cauchy sequence if and only if $\{x_n\}$ is a Cauchy sequence in the metric space (X,d_p);*

(b) *(X,p) is a complete partial metric space if and only if (X,d_p) is a complete metric space. Moreover, $\lim_{n\to\infty} d_p(x,x_n)=0$ if and only if*

$$\lim_{n\to\infty} p(x,x_n)=\lim_{n,m\to\infty} p(x_n,x_m)=p(x,x).$$

One can easily prove the next results.

Lemma 1.2. *Let (X,p) be a partial metric space. Then*

(a) *If $p(x,y)=0$, then $x=y$;*

(b) *If $x\neq y$, then $p(x,y)>0$.*

Lemma 1.3. *Let (X,p) be a partial metric space and $x_n\to z$ with $p(z,z)=0$. Then $\lim_{n\to\infty} p(x_n,y)=p(z,y)$ for all $y\in X$.*

For a partial metric space (X,p), let $CB(X)$ denote the collection of all nonempty, bounded and closed subsets of X. Let H_p be defined on $CB(X)$ by

$$H_p(A,B)=\max\Big\{\sup_{a\in A} p(a,B),\sup_{b\in B} p(b,A)\Big\},$$

where $p(x,A):=\inf\{p(x,y): y\in A\}$. Note that $p(x,Tx)\le p(x,y)+p(y,Tx)$ for all $x,y\in X$. In fact, for each $y\in X$ and $a\in Tx$ we have

$$p(x,a)\le p(x,y)+p(y,a)-p(y,y)\le p(x,y)+p(y,a)$$

and so $p(x,Tx)=\inf_{a\in Tx} p(x,a)\le p(x,y)+\inf_{a\in Tx} p(y,a)=p(x,y)+p(y,Tx)$. In this paper, we prove some results on fixed point of multivalued operators on partial metric spaces.

2 Main results

First, we define the non-increasing function $\theta:[0,1)\to(\frac{1}{2},1)$ by

$$\theta(r)=\begin{cases}\dfrac{1}{2}, & 0\le r\le \dfrac{1}{2}(\sqrt{5}-1),\\ \dfrac{1-r}{r^2}, & \dfrac{1}{2}(\sqrt{5}-1)\le r\le \dfrac{1}{\sqrt{2}},\\ \dfrac{1}{1+r}, & \dfrac{1}{\sqrt{2}}\le r<1.\end{cases}$$

Theorem 2.1. *Let (X,p) be a complete partial metric space and $T:X\to CB(X)$ a multivalued operator. Assume that there exists $r\in[0,1)$ such that for all $x,y\in X$, $\theta(r)p(x,Tx)\le p(x,y)$ implies $H_p(Tx,Ty)\le rM(x,y)$, where*

$$M(x,y)=\max\Big\{p(x,y),\, p(x,Tx),\, \frac{r}{2}p(y,Ty),\, \frac{p(x,Ty)+p(y,Tx)}{3}\Big\}.$$

Then T has a fixed point in X.

Proof. Let $x_0\in X$ and $x_1\in Tx_0$. If $x_0=x_1$, then x_0 is a fixed point of T. Suppose that $x_1\neq x_0$. Since $\theta(r)\le 1$, $\theta(r)p(x_0,Tx_0)\le p(x_0,Tx_0)\le p(x_0,x_1)$ and so $p(x_1,Tx_1)\le H_p(Tx_0,Tx_1)\le rM(x_0,x_1)$, where

$$M(x_0,x_1)=\max\Big\{p(x_0,x_1),p(x_0,Tx_0),\frac{r}{2}p(x_1,Tx_1),\frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{3}\Big\}.$$

If $M(x_0,x_1)=\frac{r}{2}p(x_1,Tx_1)$, then $p(x_1,Tx_1)\le\frac{r^2}{2}p(x_1,Tx_1)<p(x_1,Tx_1)$ which is a contradiction. Thus we should have

$$M(x_0,x_1)=\max\Big\{p(x_0,x_1),\, \frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{3}\Big\}.$$

If $M(x_0,x_1)=p(x_0,x_1)$, then $p(x_1,Tx_1)\le H_p(Tx_0,Tx_1)\le rp(x_0,x_1)$.

If $M(x_0,x_1)=\frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{3}$, then

$$M(x_0,x_1)\le\frac{p(x_0,Tx_1)+p(x_1,x_1)}{3}\le\frac{p(x_0,x_1)+p(x_1,Tx_1)}{3}.$$

Hence

$$p(x_1,Tx_1)\le H_p(Tx_0,Tx_1)\le r\frac{p(x_0,x_1)+p(x_1,Tx_1)}{3}\le\frac{2r}{3}p(x_0,x_1)+\frac{1}{2}p(x_1,Tx_1).$$

Thus $p(x_1,Tx_1)\le rp(x_0,x_1)$ and so there exists $x_2\in Tx_1$ such that

$$p(x_1,x_2)\le rp(x_0,x_1).$$

By continuing this process, we obtain a sequence $\{x_n\}_{n\geq 0}$ in X such that $x_{n+1}\in Tx_n$ and $p(x_n,x_{n+1})\leq r^n p(x_0,x_1)$ for all $n\geq 0$. Now, for each $n<m$ we have

$$\begin{aligned}p(x_n,x_m)&\leq\sum_{i=n}^{m-1}p(x_i,x_{i+1})-\sum_{i=n}^{m-1}p(x_i,x_i)\leq\sum_{i=n}^{m-1}p(x_i,x_{i+1})\\&\leq\sum_{i=n}^{m-1}r^i p(x_0,x_1)\leq\frac{r^n}{1-r}p(x_0,x_1).\end{aligned}$$

Hence $\lim_{n,m\to\infty}p(x_n,x_m)=0$, that is, $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in (X,p). Now by Lemma 1.1, $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in (X,d_p). Since (X,p) is complete, so is (X,d_p). Thus, there exists $z\in X$ such that $x_n\to z$ in (X,d_p). Moreover, by Lemma 1.1 we have

$$p(z,z)=\lim_{n\to\infty}p(x_n,x)=\lim_{n,m\to\infty}p(x_n,x_m)=0.$$

Now, we show that $p(z,Tx)\leq rp(z,x)$ for all $x\in X\setminus\{z\}$. Let $x\in X\setminus\{z\}$. Since $x_n\to z$ and $p(z,z)=0$, there exists a natural number n_0 such that $p(z,x_n)\leq\frac{1}{3}p(x,z)$ for all $n\geq n_0$. Then, we have

$$\begin{aligned}\theta(r)p(x_n,Tx_n)&\leq p(x_n,Tx_n)\leq p(x_n,x_{n+1})\leq p(x_n,z)+p(z,x_{n+1})-p(z,z)\\&=p(x_n,z)+p(z,x_{n+1})\leq\frac{2}{3}p(x,z)=p(x,z)-\frac{1}{3}p(x,z)\\&\leq p(x,z)-p(z,x_n)\leq p(x,x_n).\end{aligned}$$

Hence $H_p(Tx_n,Tx)\leq rM(x_n,x)$, where

$$M(x_n,x)=\max\Big\{p(x_n,x),\,p(x_n,Tx_n),\,\frac{r}{2}p(x,Tx),\,\frac{p(x,Tx_n)+p(x_n,Tx)}{3}\Big\}.$$

It follows that $p(x_{n+1},Tx)\leq rM(x_n,x)$ for all $n\geq n_0$. If $M(x_n,x)=p(x_n,x)$ or $M(x_n,x)=p(x_n,Tx_n)$, then we obtain $p(x_{n+1},Tx)\leq rp(x_n,x)$ and so

$$\lim_{n\to\infty}p(x_{n+1},Tx)\leq r\lim_{n\to\infty}p(x_n,x)\ \Rightarrow\ p(z,Tx)\leq rp(z,x).$$

If $M(x_n,x)=\frac{r}{2}p(x,Tx)$, then

$$\begin{aligned}p(x_{n+1},Tx)\leq\frac{r^2}{2}p(x,Tx)&\leq\frac{r^2}{2}[p(x,x_n)+p(x_n,Tx)-p(x_n,x_n)]\\&\leq\frac{r^2}{2}[p(x,x_n)+p(x_n,Tx)].\end{aligned}$$

If $0\leq r\leq\frac{1}{2}(\sqrt{5}-1)$, then $r<1-r^2$ and so

$$\lim_{n\to\infty}p(x_{n+1},Tx)\leq r^2\lim_{n\to\infty}[p(x,x_n)+p(x_n,Tx)].$$

Hence

$$rp(z,Tx)\le(1-r^2)p(z,Tx)\le r^2p(z,x)\ \Rightarrow\ p(z,Tx)\le rp(z,x).$$

If $\frac{1}{2}(\sqrt{5}-1)\le r\le\frac{1}{\sqrt{2}}$ or $\frac{1}{\sqrt{2}}\le r<1$, then $1<2-r^2$ and so

$$\lim_{n\to\infty}p(x_{n+1},Tx)\le\frac{r^2}{2}\lim_{n\to\infty}[p(x,x_n)+p(x_n,Tx)],$$
$$\Big(1-\frac{r^2}{2}\Big)p(z,Tx)\le r^2p(z,x)\le rp(z,x).$$

Hence, $p(z,Tx)\le(2-r^2)p(z,Tx)\le rp(z,x)$. Therefore, $p(z,Tx)\le rp(z,x)$ holds for all $x\in X\setminus\{z\}$. Now, we show that $H_p(Tx,Tz)\le rM(x,z)$ for all $x\neq z$, where

$$M(x,z)=\max\Big\{p(z,x),p(x,Tx),\frac{r}{2}p(z,Tz),\frac{p(x,Tz)+p(z,Tx)}{3}\Big\}.$$

Let $x\neq z$. If $0\le r\le\frac{1}{2}(\sqrt{5}-1)$, then $\theta(r)=\frac{1}{2}$ and so

$$\begin{aligned}p(x,Tx)&\le p(x,z)+p(z,Tx)-p(z,z)=p(x,z)+p(z,Tx)\\&\le(1+r)p(x,z)\le 2p(x,z).\end{aligned}$$

Hence $\theta(r)p(x,Tx)\le p(x,z)$ and so $H_p(Tx,Tz)\le rM(x,z)$. If $\frac{1}{2}(\sqrt{5}-1)\le r\le\frac{1}{\sqrt{2}}$, then $\theta(r)=\frac{1-r}{r^2}$ and so for each $n\ge1$, there exists $y_n\in Tx$ such that

$$p(z,y_n)\le p(z,Tx)+\Big(r+\frac{1}{n}\Big)p(z,x).$$

Thus

$$\begin{aligned}p(x,Tx)&\le p(x,y_n)\le p(x,z)+p(z,y_n)-p(z,z)\\&=p(x,z)+p(z,y_n)\le p(x,z)+p(z,Tx)+\Big(r+\frac{1}{n}\Big)p(z,x)\\&\le p(x,z)+rp(x,z)+\Big(r+\frac{1}{n}\Big)p(z,x)=\Big(1+2r+\frac{1}{n}\Big)p(z,x)\end{aligned}$$

for all $n\ge1$. Hence, $\theta(r)p(x,Tx)\le\frac{1}{1+2r}p(x,Tx)\le p(z,x)$ and so

$$H_p(Tx,Tz)\le rM(x,z).$$

If $\frac{1}{\sqrt{2}}\le r<1$, then $\theta(r)=\frac{1}{1+r}$ and so

$$p(x,Tx)\le p(x,z)+p(z,Tx)-p(z,z)\le(1+r)p(x,z).$$

Hence $\theta(r)p(x,Tx)=\frac{1}{1+r}p(x,Tx)\le p(z,x)$ and so $H_p(Tx,Tz)\le rM(x,z)$. Now put $\mathcal{A}=\{n\ge1:x_n=z\}$. First suppose that $\mathcal{A}$ is finite. In this case, by abandon the indices in $\mathcal{A}$, we have

$$p(z,Tz)=\lim_{n\to\infty}p(x_{n+1},Tz)\le\lim_{n\to\infty}H_p(Tx_n,Tz)\le\lim_{n\to\infty}rM(x_n,z),$$

where

$$M(x_n,z)=\max\Big\{p(x_n,z),\,p(x_n,Tx_n),\,\frac{r}{2}p(z,Tz),\,\frac{p(x_n,Tz)+p(z,Tx_n)}{3}\Big\}.$$

If $M(x_n,z)=p(x_n,z)$, then

$$p(z,Tz)\le\lim_{n\to\infty}rp(x_n,z)=rp(z,z)=0.$$

If $M(x_n,z)=p(x_n,Tx_n)$, then

$$p(z,Tz)\le\lim_{n\to\infty}rp(x_n,Tx_n)\le\lim_{n\to\infty}rp(x_n,x_{n+1})=0.$$

If $M(x_n,z)=\frac{r}{2}p(z,Tz)$, then

$$p(z,Tz)\le\lim_{n\to\infty}\frac{r^2}{2}p(z,Tz)=\frac{r^2}{2}p(z,Tz).$$

If $M(x_n,z)=\frac{p(x_n,Tz)+p(z,Tx_n)}{3}$, then

$$p(z,Tz)\le\lim_{n\to\infty}r\frac{p(x_n,Tz)+p(z,Tx_n)}{3}\le\lim_{n\to\infty}r\frac{p(x_n,Tz)+p(z,x_{n+1})}{3}=\frac{r}{3}p(z,Tz).$$

Hence $p(z,Tz)=0$. Now, suppose that $\mathcal{A}$ is an infinite set. Let $\{x_n\}_{n\in\mathcal{A}}=\{x_{n_k}\}_{k\ge1}$. In this case, for each $k\ge1$ we have $p(z,Tz)=p(z,Tx_{n_k})\le p(z,x_{n_k+1})$. Thus $p(z,Tz)\le\lim_{k\to\infty}p(z,x_{n_k+1})=0$. Hence $p(z,Tz)=0$. Therefore in each case we have $p(z,Tz)=0$ and so $z\in\overline{Tz}=Tz$. □

The following are some examples for Theorem 2.1.

Example 2.1. Let $X=[0,1]$, $p(x,y)=\max\{x,y\}$ and $T:X\to2^X$ be defined by $Tx=\{\frac{1}{3}x^2\}$ for all $x\in X$. It is easy to check that (X,p) is a complete partial metric space and for $r=\frac{1}{3}$, $\theta(r)p(x,Tx)\le p(x,y)$ implies $H_p(Tx,Ty)\le rM(x,y)$ for all $x,y\in X$.

Recently, Haghi, Rezapour and Shahzad showed that researchers should be careful about obtaining fixed point results on partial metric spaces [3]. Next example show that our result does not satisfy in the process of [3].

Example 2.2. Let $X=[0,\infty)$, $p(x,y)=\max\{x,y\}$, $r\in(0,1)$ and $T_r:X\to2^X$ be defined by $T_rx=\{0,rx\}$ for all $x\in X$. It is easy to check that (X,p) is a complete partial metric space and $\theta(r)p(x,Tx)\le p(x,y)$ implies $H_p(Tx,Ty)\le rM(x,y)$ for all $x,y\in X$.

Theorem 2.2. *Let (X,d) be a complete partial metric space and $T:X\to CB(X)$ a multivalued operator. Suppose that $\varphi:[0,\infty)\to[0,\infty)$ is an increasing sublinear function such that $\varphi(0)=0$, $\varphi(t)<t$ and $\sum_{n=1}^{\infty}\varphi^n(t)<\infty$ for all $t\ge0$. Also, suppose that $\theta(r)p(x,Tx)\le p(x,y)$ implies $H_p(Tx,Ty)\le r\varphi(M(x,y))$ for all $x,y\in X$, where*

$$M(x,y)=\max\Big\{p(x,y),\,p(x,Tx),\,\frac{1}{2}p(y,Ty),\,\frac{p(x,Ty)+p(y,Tx)}{3}\Big\}.$$

Then T has a fixed point in X.

Proof. Let $x_0\in X$ and $x_1\in Tx_0$. If $x_0=x_1$, then x_0 is a fixed point of T. Suppose that $x_1\neq x_0$. Since $\theta(r)\le 1$, $\theta(r)p(x_0,Tx_0)\le p(x_0,Tx_0)\le p(x_0,x_1)$ and so $p(x_1,Tx_1)\le H_p(Tx_0,Tx_1)\le r\varphi(M(x_0,x_1))$, where

$$M(x_0,x_1)=\max\Big\{p(x_0,x_1),p(x_0,Tx_0),\frac{1}{2}p(x_1,Tx_1),\frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{3}\Big\}.$$

If $x_1\in Tx_1$, then x_1 is a fixed point of T. Let $x_1\notin Tx_1$. Then $p(x_1,Tx_1)>0$ and so $M(x_0,x_1)\neq \frac{1}{2}p(x_1,Tx_1)$. Thus,

$$M(x_0,x_1)=\max\Big\{p(x_0,x_1),\frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{3}\Big\}.$$

If $M(x_0,x_1)=p(x_0,x_1)$, then $p(x_1,Tx_1)\le H_p(Tx_0,Tx_1)\le r\varphi(p(x_0,x_1))$.

If $M(x_0,x_1)=\frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{3}$, then

$$M(x_0,x_1)\le\frac{p(x_0,Tx_1)+p(x_1,x_1)}{3}\le\frac{p(x_0,x_1)+p(x_1,Tx_1)}{3}$$

and so

$$\begin{aligned}p(x_1,Tx_1)&\le H_p(Tx_0,Tx_1)\le r\varphi\Big(\frac{p(x_0,x_1)+p(x_1,Tx_1)}{3}\Big)\\&\le\frac{2r}{3}\varphi(p(x_0,x_1))+\frac{1}{3}\varphi(p(x_1,Tx_1)).\end{aligned}$$

Hence in each case we obtain $p(x_1,Tx_1)\le r\varphi(p(x_0,x_1))$. This implies that there exists $x_2\in Tx_1$ such that $p(x_1,x_2)\le r\varphi(p(x_0,x_1))$. By continuing this process, we obtain a sequence $\{x_n\}_{n\ge 0}$ in X such that $p(x_n,x_{n+1})\le r^n\varphi^n(p(x_0,x_1))$ and $x_{n+1}\in Tx_n$ for all $n\ge 0$. Now, for $n<m$ we have

$$p(x_n,x_m)\le\sum_{i=n}^{m-1}p(x_i,x_{i+1})-\sum_{i=n}^{m-1}p(x_i,x_i)\le\sum_{i=n}^{m-1}p(x_i,x_{i+1})\le\sum_{i=n}^{m-1}r^i\varphi^i(p(x_0,x_1).$$

Hence $\lim_{n,m\to\infty}p(x_n,x_m)=0$, that is, $\{x_n\}_{n\ge 0}$ is a Cauchy sequence in (X,p). Now by Lemma 1.1, $\{x_n\}_{n\ge 0}$ is a Cauchy sequence in (X,d_p). Since (X,p) is complete, (X,d_p) so is. Thus there exists $z\in X$ such that $x_n\to z$ in (X,d_p). Moreover, by Lemma 1.1 we have $p(z,z)=\lim_{n\to\infty}p(x_n,x)=\lim_{n,m\to\infty}p(x_n,x_m)=0$. Now, we show that $p(z,Tx)\le r\varphi(p(z,x))$ for all $x\in X\backslash\{z\}$. Let $x\in X\backslash\{z\}$. Since $x_n\to z$ and $p(z,z)=0$, there exists a natural number n_0 such that $p(z,x_n)\le\frac{1}{3}p(x,z)$ for all $n\ge n_0$. Then we have

$$\begin{aligned}\theta(r)p(x_n,Tx_n)&\le p(x_n,Tx_n)\le p(x_n,x_{n+1})\le p(x_n,z)+p(z,x_{n+1})-p(z,z)\\&=p(x_n,z)+p(z,x_{n+1})\le\frac{2}{3}p(x,z)=p(x,z)-\frac{1}{3}p(x,z)\\&\le p(x,z)-p(z,x_n)\le p(x,x_n).\end{aligned}$$

Hence $H_p(Tx_n,Tx)\leq r\varphi(M(x_n,x))$, where

$$M(x_n,x)=\max\Big\{p(x_n,x),\, p(x_n,Tx_n),\, \frac{1}{2}p(x,Tx),\, \frac{p(x,Tx_n)+p(x_n,Tx)}{3}\Big\}.$$

It follows that $p(x_{n+1},Tx)\leq r\varphi(M(x_n,x))$ for all $n\geq n_0$. If $M(x_n,x)=p(x_n,x)$ or $M(x_n,x)=p(x_n,Tx_n)$, then $p(x_{n+1},Tx)\leq r\varphi(p(x_n,x))$ and so

$$p(z,Tx)\leq r\varphi(p(z,x)).$$

If $M(x_n,x)=\frac{1}{2}p(x,Tx)$, then

$$\begin{aligned}p(x_{n+1},Tx)&\leq \frac{r}{2}\varphi(p(x,Tx))\leq \frac{r}{2}\varphi(p(x,x_n)+p(x_n,Tx)-p(x_n,x_n))\\&\leq \frac{r}{2}\varphi(p(x,x_n))+\frac{1}{2}\varphi(p(x_n,Tx)).\end{aligned}$$

Hence $p(z,Tx)\leq \frac{r}{2}\varphi(p(x,z))+\frac{1}{2}\varphi(p(z,Tx))$ and so $p(z,Tx)\leq r\varphi(p(x,z))$. If $M(x_n,x)=\frac{p(x,Tx_n)+p(x_n,Tx)}{3}$, then

$$p(x_{n+1},Tx)\leq r\varphi\Big(\frac{p(x,Tx_n)+p(x_n,Tx)}{3}\Big)\leq \frac{r}{3}\varphi(p(x,x_{n+1}))+\frac{1}{3}\varphi(p(x_n,Tx)).$$

Hence $p(z,Tx)\leq \frac{r}{3}\varphi(p(x,z))+\frac{1}{3}p(z,Tx)$ and so $p(z,Tx)\leq r\varphi(p(x,z))$. Now, we show that $H_p(Tx,Tz)\leq r\varphi(M(x,z))$ for all $x\neq z$, where

$$M(x,z)=\max\Big\{p(z,x),\, p(x,Tx),\, \frac{1}{2}p(z,Tz),\, \frac{p(x,Tz)+p(z,Tx)}{3}\Big\}.$$

Let $x\neq z$. If $0\leq r\leq \frac{1}{2}(\sqrt{5}-1)$, then $\theta(r)=\frac{1}{2}$, and so

$$\begin{aligned}p(x,Tx)&\leq p(x,z)+p(z,Tx)-p(z,z)=p(x,z)+p(z,Tx)\\&\leq p(x,z)+r\varphi(p(x,z))\leq (1+r)p(x,z)\leq 2p(x,z).\end{aligned}$$

Hence $\theta(r)p(x,Tx)\leq p(x,z)$ and so $H_p(Tx,Tz)\leq rM(x,z)$. If $\frac{1}{2}(\sqrt{5}-1)\leq r\leq \frac{1}{\sqrt{2}}$, then $\theta(r)=\frac{1-r}{r^2}$ and for each $n\geq 1$ there exists $y_n\in Tx$ such that

$$p(z,y_n)\leq p(z,Tx)+\Big(r+\frac{1}{n}\Big)p(z,x).$$

Thus

$$\begin{aligned}p(x,Tx)&\leq p(x,y_n)\leq p(x,z)+p(z,y_n)-p(z,z)\\&=p(x,z)+p(z,y_n)\leq p(x,z)+p(z,Tx)+\Big(r+\frac{1}{n}\Big)p(z,x)\\&\leq p(x,z)+r\varphi(p(x,z))+\Big(r+\frac{1}{n}\Big)p(z,x)<\Big(1+2r+\frac{1}{n}\Big)p(z,x)\end{aligned}$$

for all $n\geq 1$. Hence $\theta(r)p(x,Tx)\leq \frac{1}{1+2r}p(x,Tx)\leq p(z,x)$ and so

$$H_p(Tx,Tz)\leq r\varphi(M(x,z)).$$

If $\frac{1}{\sqrt{2}}\leq r<1$, then $\theta(r)=\frac{1}{1+r}$ and so

$$p(x,Tx)\leq p(x,z)+p(z,Tx)-p(z,z)\leq p(x,z)+r\varphi(p(z,x))\leq (1+r)p(x,z).$$

Hence, $\theta(r)p(x,Tx)=\frac{1}{1+r}p(x,Tx)\leq p(z,x)$ and so $H_p(Tx,Tz)\leq r\varphi(M(x,z))$. Now, put $\mathcal{A}=\{n\geq 1: x_n=z\}$. First, suppose that $\mathcal{A}$ is finite. In this case, by abandon the indices in $\mathcal{A}$, we have

$$p(z,Tz)=\lim_{n\to\infty}p(x_{n+1},Tz)\leq \lim_{n\to\infty}H_p(Tx_n,Tz)\leq \lim_{n\to\infty}r\varphi(M(x_n,z)),$$

where

$$M(x_n,z)=\max\Big\{p(x_n,z),p(x_n,Tx_n),\frac{1}{2}p(z,Tz),\frac{p(x_n,Tz)+p(z,Tx_n)}{3}\Big\}.$$

If $M(x_n,z)=p(x_n,z)$, then

$$p(z,Tz)\leq \lim_{n\to\infty}r\varphi(p(x_n,z))=r\varphi(p(z,z))=0.$$

If $M(x_n,z)=p(x_n,Tx_n)$, then

$$p(z,Tz)\leq \lim_{n\to\infty}rp(x_n,Tx_n)\leq \lim_{n\to\infty}r\varphi(p(x_n,x_{n+1}))=0.$$

If $M(x_n,z)=\frac{1}{2}p(z,Tz)$, then

$$p(z,Tz)\leq \lim_{n\to\infty}\frac{r}{2}\varphi(p(z,Tz))=\frac{r}{2}\varphi(p(z,Tz)).$$

If $M(x_n,z)=\frac{p(x_n,Tz)+p(z,Tx_n)}{3}$, then

$$\begin{aligned}p(z,Tz)&\leq \lim_{n\to\infty}r\varphi\Big(\frac{p(x_n,Tz)+p(z,Tx_n)}{3}\Big)\\&\leq \lim_{n\to\infty}\frac{r}{3}\varphi(p(x_n,Tz))+\frac{1}{3}\varphi(p(z,x_{n+1}))=\frac{r}{3}\varphi(p(z,Tz)).\end{aligned}$$

Hence $p(z,Tz)=0$. Now, suppose that $\mathcal{A}$ is an infinite set. Let $\{x_n\}_{n\in\mathcal{A}}=\{x_{n_k}\}_{k\geq 1}$. In this case, for each $k\geq 1$ we have $p(z,Tz)=p(z,Tx_{n_k})\leq p(z,x_{n_k+1})$. Thus, $p(z,Tz)\leq \lim_{k\to\infty}p(z,x_{n_k+1})=0$. Hence $p(z,Tz)=0$. Therefore, in each case we have $p(z,Tz)=0$ and so $z\in\overline{Tz}=Tz$. □

By using a similar proof, we can provide the next result.

Theorem 2.3. *Let (X,p) be a complete partial metric space, $T:X\to CB(X)$ a multivalued operator and $F:[0,\infty)\to[0,\infty)$ an increasing subadditive continuous function such that $F(0)=0$ and $F(t)>0$ for all $t\in[0,\infty)$. Also, suppose that $\theta(r)F(p(x,Tx))\le F(p(x,y))$ implies $F(H_p(Tx,Ty))\le rF(M(x,y))$ for all $x,y\in X$, where*

$$M(x,y)=\max\Big\{p(x,y),\,p(x,Tx),\,\frac{r}{2}p(y,Ty),\,\frac{p(x,Ty)+p(y,Tx)}{3}\Big\}.$$

Then T has a fixed point.

Corollary 2.1. *Let (X,d) be a complete partial metric space, $T:X\to CB(X)$ a multivalued operator and $\phi:[0,\infty)\to[0,\infty)$ an integrable function such that*

$$\int_0^{\varepsilon}\phi(t)\mathrm{d}t>0$$

for all $t\ge 0$ and $\varepsilon>0$. Suppose that

$$\int_0^{p(x,Tx)}\phi(t)dt\le\int_0^{p(x,y)}\phi(t)\mathrm{d}t$$

implies

$$\int_0^{H_p(Tx,Ty)}\phi(t)\mathrm{d}t\le r\int_0^{M(x,y)}\phi(t)\mathrm{d}t$$

for all $x,y\in X$, where

$$M(x,y)=\max\Big\{p(x,y),\,p(x,Tx),\,\frac{r}{2}p(y,Ty),\,\frac{p(x,Ty)+p(y,Tx)}{3}\Big\}.$$

Then T has a fixed point.

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